Mathematical description of built-up structure I

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1 Motivation
   - Urban structure

2 Mathematical formulation
   - Basic definitions
   - Example
   - Properties
   - Ergodicity
   - Covariance

3 Conclusion
Analysis of urban structure

Focus on a pattern given by building footprints in the centre of cities

Available datasets

- Building footprints in the USA
- Cadastral records in the Czech Republic
Building footprints in the USA
Corresponding buildings
Cadastral records in the Czech Republic
Cadastral records in the Czech Republic
Two natural representation of the data

**Point pattern**
Buildings as points located at $r_1, r_2, \ldots$

**Set pattern**
Buildings as polygons (2D objects) - closed sets
Two natural representation of the data

**Point pattern**
Buildings as points located at $r_1, r_2, \ldots$

**Set pattern**
Buildings as polygons (2D objects) - closed sets

Suitable mathematical description - **Stochastic geometry**
Street pattern

- Streets correlated with buildings
- Representation as a fibre process (pattern)
Main object - Random Closed Set

Let $\mathcal{F}$ resp. $\mathcal{G}$ resp. $\mathcal{K}$ denote the class of all closed resp. open resp. compact subsets of $\mathbb{R}^d$ (generally LCHS space).

For each $A \subset \mathbb{R}^d$ we define

$$\mathcal{F}_A = \{ F \in \mathcal{F} | F \cap A \neq \emptyset \},$$

as the family of closed sets which hits $A$ and

$$\mathcal{F}^A = \{ F \in \mathcal{F} | F \cap A = \emptyset \},$$

as the family of closed sets which miss $A$.

Let

$$\mathbb{B} = \{ \mathcal{F}_{G_1,G_2,\ldots,G_n}^K | K \in \mathcal{K}, G_i \in \mathcal{G}, n \in \mathbb{N}_0 \}$$

where

$$\mathcal{F}_{G_1,G_2,\ldots,G_n}^K = \mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \ldots \cap \mathcal{F}_{G_n}, \ n > 0.$$ 

The topology generated by the base $\mathbb{B}$ is called hit-or-miss topology on $\mathcal{F}$. 
Random closed sets

- The Borel σ-algebra generated by the hit-or-miss topology is denoted as $\mathcal{B}(\mathcal{F})$.
- Note that $\mathcal{B}(\mathcal{F})$ particularly contains $\mathcal{F}^K$, $\mathcal{F}_K$ and $\mathcal{F}^G$, $\mathcal{F}_G$ for all $K$ compact and $G$ open.

**Definition**

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$ be a measurable space introduced above. The map

$$X : \Omega \to \mathcal{F} \quad (1)$$

is called *random closed set* if $X$ is $\mathcal{A} - \mathcal{B}(\mathcal{F})$ measurable, i.e.

$$X^{-1}(F) = \{ \omega \in \Omega | X(\omega) \in F \} \in \mathcal{A} \quad (2)$$

for each $F \in \mathcal{B}(\mathcal{F})$.

Distribution of $X$ is uniquely defined by the *capacity functional* $T_X : \mathcal{K} \to [0, 1]$ defined by $T_X(K) = P(\mathcal{F}_K) = P(X \cap K \neq \emptyset)$, $K \in \mathcal{K}$ - Choquet theorem.
Random measures

- A measure $\mu$ on the family $\mathcal{B}$ of Borel sets in $\mathbb{R}^d$ is said to be *locally finite* if $\mu$ is finite on bounded subsets of $\mathbb{R}^d$.

- The set $\mathcal{M}$ of all locally finite measures on $\mathcal{B}$ can be endowed with a $\sigma$-algebra $\mathcal{B}(\mathcal{M})$ generated as the smallest $\sigma$-algebra that contains subsets $\{\mu \in \mathcal{M} | \mu(B) > t\}$ for all $t > 0$ and $B \in \mathcal{B}$.

**Definition**

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ be a measurable space introduced above. The map

$$ M : \Omega \rightarrow \mathcal{M} $$

is called *random measure* if $M$ is $\mathcal{A} - \mathcal{B}(\mathcal{M})$ measurable, i.e.

$$ M^{-1}(Y) = \{ \omega \in \Omega | M(\omega) \in Y \} \in \mathcal{A} $$

for each $Y \in \mathcal{B}(\mathcal{M})$. 
Let $X$ be a random closed set and $\mu$ be a fixed measure on $\mathbb{R}^d$ (e.g. Lebesgue measure) such that $\mu(B \cap X)$ is almost surely finite for every bounded Borel set $B \in \mathcal{B}$. Then

$$M_X(B) = \mu(B \cap X)$$

is a locally finite random measure.

The support of a measure $\mu$ is defined as a set

$$\text{supp } \mu = \{ x \in \mathbb{R}^d | \forall G \in \mathcal{G}, x \in G \Rightarrow \mu(G) > 0 \}.$$
A measure $\mu$ on the family of $B$ of Borel sets in $\mathbb{R}^d$ is said to be *counting* if it takes only non-negative integer values. A counting measure $\mu$ is *locally finite* if $\mu$ is finite on bounded subsets of $\mathbb{R}^d$.

The set $N$ of all locally finite counting measures on $B$ can be endowed with a $\sigma$-algebra $\mathcal{B}(N)$ generated as the smallest $\sigma$-algebra that contains subsets $\{\mu \in N | \mu(B) = k\}$ for all $k \in \mathbb{N}_0$ and $B \in B$.

**Definition**

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(N, \mathcal{B}(N))$ be a measurable space introduced above. The map

$$\Pi : \Omega \rightarrow N$$

(5)

is called *point process* if $\Pi$ is $\mathcal{A} - \mathcal{B}(N)$ measurable, i.e.

$$\Pi^{-1}(N) = \{\omega \in \Omega | \Pi(\omega) \in N\} \in \mathcal{A}$$

(6)

for each $N \in \mathcal{B}(N)$. 

Daniel Vašata (FNSPE)
A point process is called \textit{simple} if \(\sup_{x \in \mathbb{R}^d} \Pi(\{x\}) \leq 1\), almost surely.

**Proposition**

The map \(\Pi : \Omega \rightarrow \mathcal{N}\) is a simple point process if and only if its support \(\text{supp} \: \Pi\) is a locally finite random closed set.
A point process is called \textit{simple} if \( \sup_{x \in \mathbb{R}^d} \Pi(\{x\}) \leq 1 \), almost surely.

**Proposition**

The map \( \Pi : \Omega \to \mathcal{N} \) is a simple point process if and only if its support \( \text{supp} \, \Pi \) is a locally finite random closed set.

- The mean value \( \Lambda \) of a random measure \( M \) is called the \textit{intensity measure}:
  \[
  \Lambda(B) = E[M(B)], \quad B \in \mathcal{B}
  \]

- Intensity measure not necessarily locally finite or \( \sigma \)-finite

- If \( \Lambda \) is absolutely continuous with respect to the Lebesgue measure, \( \Lambda \ll \nu_d \), then the corresponding Radon-Nikodym derivative \( \lambda \) is called the \textit{intensity function} of \( M \).

- For the random closed set \( X \) the intensity function exists and
  \[
  \lambda(x) = E[\mathbb{1}_X(x)]
  \]
The **Poisson point process** with intensity measure \( \Lambda \) is a point process \( \Pi \) on \( \mathbb{R}^d \) such that

- for every bounded Borel set \( B \in \mathcal{B} \), the count \( \Pi(B) \) has a Poisson distribution with mean \( \Lambda(B) \)
- if \( B_1, \ldots, B_n \) are disjoint bounded sets, then \( \Pi(B_1), \ldots, \Pi(B_n) \) are independent

For intensity measure with intensity function \( \lambda(x) \)

\[
P(\Pi(B) = k) = \frac{\left( \int_B \lambda(x) \, dx \right)^k}{k!} \exp \left( -\int_B \lambda(x) \, dx \right)
\]

- **Ideal gas** - \( \lambda(x) = \rho = e^{\frac{\mu}{k_B T}} (\pi k_B T 2m)^{3/2} \)
Properties of random closed sets

- If two random closed sets $X$ and $Y$ follow the same distribution, that is $P(X \in F) = P(Y \in F)$ for every $F \in \mathcal{B}(\mathcal{F})$, then we write $X \overset{d}{\sim} Y$.

- **Stationarity:** $X \overset{d}{\sim} (X + y)$ for all $y \in \mathbb{R}^d$
  \[ \iff T_X(K) = T_X(K + y). \]

- **Isotropy:** $X \overset{d}{\sim} g(X)$ for each rotation $g$ of $\mathbb{R}^d$
  \[ \iff T_X(K) = T_X(g(K)). \]

- **Self-similarity:** $X \overset{d}{\sim} cX$ for every $c > 0$
  \[ \iff T_X(K) = T_X(cK). \]

Poisson point process with constant intensity function is stationary and isotropic.
Ergodicity of random closed sets

- Let denote $F_x = \{A_x | A \in F\}$ for $F \in \mathcal{B}(\mathcal{F})$ and $x \in \mathbb{R}^d$.
- A random closed set $X$ is said to be **ergodic** if the condition
  
  $$P(F \setminus F_x \cup F_x \setminus F) = 0, \ \forall x \in \mathbb{R}^d$$

  on $F \in \mathcal{B}(\mathcal{F})$ implies that $P(F) = 0$ or $P(F) = 1$.
- The poisson process with constant intensity function $\lambda(x) = \rho$ is ergodic.
- Let $C_0$ denote the half-open cube,
  
  $$C_0 = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{2}, \frac{1}{2}\right) \subset \mathbb{R}^d.$$
Ergodicity of random closed sets

**Theorem**

Suppose the mapping \( h : \mathcal{B}_0 \to \mathbb{R} \) is a measurable, translation invariant, additive set-function defined on bounded Borel sets. Suppose, moreover, that \( X \) is a stationary ergodic random closed set. Then

1. For every \( \epsilon > 0 \) there is a \( \eta = \eta(\epsilon) > 0 \) such that if

\[
\sup \left\{ E \left[ |h(X \cap K)| \right] | K \neq \emptyset, \ K \in \mathcal{C}_0 \right\} < +\infty
\]

then

\[
E \left[ \left| \frac{h(X \cap K)}{\nu_d(K)} - E[h(X \cap C_0)] \right| \right] < \epsilon
\]

for compact \( K \) with \( r(K) > \eta \).

2. Suppose there is a non-negative random variable \( \xi \) of finite mean such that \( |h(X \cap K)| < \xi \) almost surely for all non-empty \( K \in \mathcal{C}_0 \). Then

\[
\lim_{n \to +\infty} \frac{h(X \cap K_n)}{\nu_d(K_n)} = E[h(X \cap C_0)] \quad \text{almost surely.}
\]
Covariance of the stationary random closed set $X$ can be defined as

$$C(x, y) = P(x \in X, y \in Y) = E[\mathbb{1}_X(x)\mathbb{1}_X(y)].$$

Another possibility - as a covariance of random field $\mathbb{1}_X$

$$C_2(x, y) = E[(\mathbb{1}_X(x) - \lambda)(\mathbb{1}_X(y) - \lambda)] = C(x, y) - \lambda^2.$$

For the point process $\Pi$ taken as the random closed set $\text{supp} \, \Pi$

$$C(x, y) = 0, \text{ almost everywhere on } \mathbb{R}^{2d}.$$

⇒ The correlation for point process should be defined in a different way.
Moment measure

- Let take two measurable spaces \((X, \mathcal{X})\) and \((Y, \mathcal{Y})\). The natural \(\sigma\)-algebra \(\mathcal{X} \times \mathcal{Y}\) for the product space \(X \times Y\) is generated by the collection of sets \(
\{A \times B | A \in \mathcal{X}, B \in \mathcal{Y}\}\).

Definition

Let \(M\) be a random measure on the family \(\mathcal{B}\) of Borel sets in \(\mathbb{R}^d\). Then a measure \(\mu^{(n)}\) on \(\mathcal{B}^n = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}\), the family of Borel sets in \(\mathbb{R}^{dn}\), defined by

\[
\mu^{(n)}(B_1 \times B_2 \times \ldots B_n) = E \left[ M(B_1)M(B_2)\ldots M(B_n) \right] \tag{10}
\]

is called the \(n\)-th moment measure of \(M\).

- The first moment measure \(\mu^{(1)}\) is the same as the intensity measure \(\Lambda\) of \(M\).
- If the second moment measure is absolutely continuous with respect to Lebesgue measure \(\nu_{2d}\) on \(\mathbb{R}^{2d}\), we called its Radon-Nikodym derivative \(C(x, y)\) the second moment density or the covariance. In such a case

\[
\mu^{(2)}(B) = \int_B C(x, y) dx dy
\]

for \(B \in \mathcal{B}^2\).
Covariance II

For the random closed set $X$ and corresponding random measure $M_X$ the second moment density always exists since

$$\mu^{(2)}(B) = E \int_B 1_X(x) 1_X(y) dx dy.$$  

From the Fubini’s theorem it is almost everywhere given by

$$C(x, y) = E[1_X(x) 1_X(y)] = P(x \in X, y \in X).$$
Suppose we have the simple point process $\Pi$ and corresponding locally finite random closed set $X = \text{supp} \, \Pi$.

If $\gamma(B)$ is the counting measure that returns number of points in arbitrary Borel set $B$ then

$$\Pi(B) = \gamma(B \cap X) = \int_{B \cap X} d\gamma(y) = \int_B 1_X(y) d\gamma(y) = \sum_{x \in X} 1_B(x),$$

For arbitrary Borel set $A \in \mathcal{B}^2$ in $\mathbb{R}^{2d}$ possible to write

$$\mu^{(2)}(A) = E \int_A 1_X(x) 1_X(y) d\gamma(x) d\gamma(y) = E \int_{A \cap (X \times X)} d\gamma(x) d\gamma(y).$$

This integral can be divided into the non-diagonal and the diagonal part, both non-negative. For $A \in \mathcal{B}^2$ let denote

$$\text{diag} \, A = A \cap \{(x, x) \in \mathbb{R}^{2d} | x \in \mathbb{R}^d\}.$$
Covariance IV - point process

Then for $A \in \mathcal{B}^2$ in $\mathbb{R}^{2d}$

$$
\mu^{(2)}(A) = E \int_{A \setminus \text{diag } A} 1_{x}(x)1_{x}(y)d\gamma(x)d\gamma(y) + + E \int_{\text{diag } A} 1_{x}(x)1_{x}(y)d\gamma(x)d\gamma(y).
$$

(11)

The diagonal part

$$
E \int_{\text{diag } A} 1_{x}(x)1_{x}(y)d\gamma(x)d\gamma(y) = E [\Pi(\{x|(x,x) \in \text{diag } A\})]
$$

is responsible for the fact that the covariance cannot exist.

To see this just take an arbitrary set $A \subset \mathbb{R}^d$ with $\nu_d(A) > 0$ and $E[\Pi(A)] > 0$ and let $B = \text{diag } A \times A \subset \mathbb{R}^{2d}$. Obviously $\nu_{2d}(B) = 0$. On the other side

$$
\mu^{(2)}(B) > E [\Pi(\{x|(x,x) \in \text{diag } B\})] = E[\Pi(A)] > 0
$$

which contradicts the absolute continuity $\mu^{(2)} \ll \nu_{2d}$. 
Covariance V - point process

To avoid the diagonal divergence it is useful to introduce the second factorial moment measure $\alpha^{(2)}$ given by

$$\alpha^{(2)}(A \times B) = E[\Pi(A)\Pi(B)] - E[\Pi(A \cap B)]$$

(12)

and to introduce the second factorial moment density $\rho^{(2)}(x, y)$ as its Radon-Nikodym derivative.

The obvious relation to covariance function can be formally written as

$$\rho^{(2)}(x, y) = C(x, y) - \lambda(x)\delta(x - y),$$

(13)

where $\lambda(x)$ is the intensity function of the point process $\Pi$.

Example

Poisson point process with intensity function $\lambda(x) = \lambda$.

$$\mu^{(2)}(A \times B) = E[\Pi(A)\Pi(B)] = E[\Pi(A \cap B \cup A \setminus B)\Pi(A \cap B \cup B \setminus A)] =$$

$$\cdots = \lambda^2 \nu_d(A)\nu_d(B) + \lambda \nu_d(A \cap B).$$

Therefore $\rho^{(2)}(x, y) = \lambda^2$. 
To be continued...
Thank You For Your Attention!