Feynman’s path integral and mutually unbiased bases

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The meaning of Feynman’s path integral was pointed out by G. Svetlichny(2007). He focused the essential problems into the following questions:

1. For what unitary groups $U(t)$ in $L^2(\mathbb{R}^n)$ do the position bases at times 0 and $t$ tend to mutual unbiasedness as $t \to 0$?

2. Is there a discrete version of the previous question in a finite-dimensional Hilbert space which approximately simulates the propagation of a free particle?
3. What is the information-theoretic nature of the normalization factor $A$ in the short-time propagator?

We elucidate the meaning of this question in the case of nonrelativistic particle on real line.

**Complementarity and mutually unbiased bases**

Mutually unbiased bases in Hilbert spaces of finite dimensions are closely related to the quantal notion of complementarity.

Two observables $A$ and $B$ are called complementary, if their eigenvalues are non-degenerate and any two normalized eigenvectors $|u_i\rangle$ of $A$ and $|v_j\rangle$ of $B$ satisfy

$$|\langle u_i \mid v_j \rangle| = \frac{1}{\sqrt{N}}.$$  \hspace{1cm} (1)
Then in an eigenstate $|u_i\rangle$ of $A$ all eigenvalues $b_1, \ldots, b_N$ of $B$ are measured with equal probabilities. and vice versa. This means that exact knowledge of the measured value of $A$ implies maximal uncertainty to any measured value of $B$.

According to W.K. Wootters, two orthonormal bases in an $N$-dimensional complex Hilbert space

$$\{ |u_i\rangle | i = 1, 2, \ldots, N \} \text{ and } \{ |v_j\rangle | j = 1, 2, \ldots, N \} \quad (2)$$

are called mutually unbiased, if inner products between all possible pairs of vectors taken from distinct bases have the same magnitude $1/\sqrt{N}$,

$$|\langle u_i | v_j \rangle| = \frac{1}{\sqrt{N}} \quad \text{for all } i, j \in \{1, 2, \ldots, N\}. \quad (3)$$

Thus if the system is in the state $|u_i\rangle$, then transitions to any of the states $|v_j\rangle$ have equal probabilities.
In an $N$-dimensional Hilbert space, there cannot be more than $N + 1$ mutually unbiased bases and the maximal number of $N + 1$ mutually unbiased bases is when $N$ is a power of a prime.

**Quantum mechanics in finite-dimensional Hilbert spaces**

A model for quantum mechanics in a Hilbert space of finite dimension $N$ is due to H. Weyl and J. Schwinger.

In an $N$-dimensional Hilbert space with orthonormal basis $\mathcal{B} = \{\ket{0}, \ket{1}, \ldots, \ket{N - 1}\}$ the Weyl pair of unitary operators $(Q_N, P_N)$ is defined by the relations

\[
Q_N |\rho\rangle = \omega_N^{\rho} |\rho\rangle, \quad \rho = 0, 1, \ldots, N - 1,
\]

\[
P_N |\rho\rangle = |\rho - 1 \pmod{N}\rangle,
\]

where $\omega_N = \exp\left(\frac{2\pi i}{N}\right)$. If $\mathcal{B}$ is the canonical basis of $\mathbb{C}^N$, the operators $Q_N$ and $P_N$ are represented by the matrices

\[
Q_N = \text{diag}\left(1, \omega_N, \omega_N^2, \ldots, \omega_N^{N-1}\right)
\]
and

$$P_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$  \hspace{1cm} (5)

They fulfil a commutation relation

$$P_N Q_N = \omega_N Q_N P_N$$ \hspace{1cm} (6)

analogous to the Weyl’s exponential form of Heisenberg’s commutation relations. And $P_N^N = Q_N^N = I_N$, $\omega_N^N = 1$.

The finite Heisenberg group of $N^3$ unitary $N \times N$ matrices is generated by $\omega_N$, $Q_N$ and $P_N$

$$\Pi_N = \{ \omega_N^l Q_N^j P_N^\sigma \mid l, j, \sigma = 0, 1, 2, \ldots, N - 1 \}.$$  \hspace{1cm} (7)
The geometrical picture behind the above operators is the following [0]. The cyclic group $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ is the configuration space for $N$-dimensional quantum mechanics. Elements of this periodic chain $\mathbb{Z}_N$ provide labels of the vectors of the basis $B = \{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$ with the physical interpretation that $|\rho\rangle$ is the (normalized) eigenvector of position at $\rho \in \mathbb{Z}_N$. The action of $\mathbb{Z}_N$ on $\mathbb{Z}_N$ via addition modulo $N$ is represented by unitary operators $U(\sigma) = P^\sigma_N$. The action of these discrete translations on vectors $|\rho\rangle$ from basis $B$ is given by

$$U(\sigma)|\rho\rangle = P^\sigma_N|\rho\rangle = |\rho - \sigma \pmod{N}\rangle.$$  

(8)

The discrete Fourier transformation is given by the unitary Sylvester matrix $S_N$ with elements

$$(S_N)_{k\rho} = \langle \rho | k \rangle = \frac{\omega_N^{\rho k}}{\sqrt{N}},$$

(9)
which diagonalizes the momentum operator

\[ S_N^{-1} P_N S_N = Q_N \text{ or } |k\rangle = \sum_{\rho=0}^{N-1} |\rho\rangle \langle \rho | k\rangle. \]  

(10)

\( N \times N \) approximation of quantum mechanics on the real line

An interesting approximation method in quantum mechanics was proposed by E. Husstad and T. Digernes, inspired by an idea of J. Schwinger.

They approximate quantum operators in \( L^2(\mathbb{R}) \) for one-dimensional quantum systems by \( N \times N \) matrices—operators in the Hilbert space \( l^2(Z_N) \) of finite-dimensional quantum mechanics. To this end an auxiliary factor

\[ \eta_N = \sqrt{\frac{2\pi}{N}} \]
is introduced. We need to introduce two additional dimensional quantities: length unit $a$ and the corresponding unit of linear momentum $\hbar/a$. Then the position operator is approximated by the multiplication operator in position representation

$$q_N |\rho\rangle = a\eta_N \rho |\rho\rangle,$$

(11)

and the momentum operator is approximated by the multiplication operator in momentum representation

$$p_N |k\rangle = \frac{\hbar}{a} \eta_N k |k\rangle.$$

(12)

Schwinger’s geometric idea was to identify $Z_N$ with a grid in $\mathbb{R}$. For $N$ odd, he defined a sequence of grids $L_N = \{a\eta_N \rho |\rho = 0, \pm 1, \ldots, \pm (N-1)/2\}$. In the limit $N \to \infty$ the grids are becoming denser and at the same time extending to the whole real line.
Finite-dimensional analogue of quantum free particle

was formulated by Stovicek and Tolar as a discrete Galilean evolution along a finite closed linear chain. The single-step unitary time evolution operator $C_N$ proposed there is diagonal in momentum representation

$$\langle j|C_N|k\rangle = \delta_{jk}\omega_N^{-k^2}.$$  \hspace{1cm} (13)

Transformation to position representation gives

$$ (C_N)_{\rho\sigma} = \sum_{jk} \langle \rho|j\rangle \langle j|C_N|k\rangle \langle k|\sigma \rangle = \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{-j^2} + (\rho-\sigma)j. $$ \hspace{1cm} (14)

The unitary operator $C_N$ fulfils relations

$$ C_N^{-1} Q_N C_N = \omega_N Q_N P_N^2, \quad C_N^{-1} P_N C_N = P_N. $$ \hspace{1cm} (15)

Looking at free evolution in continuous phase space, we arrive to the conclusion that operator $C_N$ should be slightly modified.
However, first consider the usual one-parameter group of unitary operators

\[ T(t) = \exp\left(-\frac{i}{\hbar \frac{p^2}{2m}}t\right), \quad t \in \mathbb{R}, \quad (16) \]

describing quantum evolution of a non-relativistic free particle of mass \( m \) on the real line. The corresponding \( N \times N \) approximation is

\[ T_N(\tau) = \exp\left(-\frac{i}{\hbar \frac{p_N^2}{2m}}\tau \varepsilon\right), \quad \tau \in \mathbb{Z}, \quad (17) \]

where we have introduced a time unit \( \varepsilon \), since dynamically defined time intervals will play a special role. Thus \( t \) shall be restricted to integer multiples \( \tau \varepsilon, \tau \in \mathbb{Z}, \) of \( \varepsilon \). The time unit \( \varepsilon \) will be chosen so that (in momentum representation)

\[ T_N(\tau)|j\rangle = \exp\left(-\frac{i}{\hbar \frac{1}{2m}}\left(\frac{\hbar}{\eta_N}j\right)^2\tau \varepsilon\right)|j\rangle = \omega_N^{-\frac{1}{2}j^2\tau}|j\rangle, \quad \tau \in \mathbb{Z}, \quad (18) \]

including an additional 1/2 factor in the exponent.
Our choice is in agreement with a dynamical relation

\[ \varepsilon = \frac{ma^2}{\hbar} \quad \text{or} \quad \frac{m}{\varepsilon} = \frac{\hbar}{a} \]  

(19)

expressing the natural fact that a particle of momentum \( \hbar/a \) traverses the distance \( a \) in time \( \varepsilon \).

One step transformation

\[ T_N(1)|j\rangle = \omega_N^{-\frac{1}{2}j^2} |j\rangle = C_{N1}|j\rangle, \]

from now on, will be denoted \( C_{N1} \). The modified unitary operator \( C_{N1} \) now fulfills relations

\[ C_{N1}^{-1}Q_N C_{N1} = \omega_N^{\frac{1}{2}} Q_N P_N, \quad C_{N1}^{-1} P_N C_{N1} = P_N, \]  

(20)
$N \times N$ approximation of the Feynman path integral

Let $|q(0), 0\rangle$ and $|q(t), t\rangle$ be the state vectors of the initial state and of the final state of a particle on $\mathbb{R}$ at times $0$, $t$, respectively. If $S[q]$ is the classical action functional of the particle, the evolution amplitude is according to Feynman formally written as a path integral.

$$
\langle q(t), t|q(0), 0\rangle = \int e^{\frac{i}{\hbar} S[q]} \mathcal{D}q(t).
$$

(21)

It is understood as a sum over all continuous paths in configuration space.

In quantum mechanics, the path integral is traditionally defined as a limit via discretization based on the division of the time interval, e.g. into $n$ intervals of equal duration $\varepsilon = t/n$. The evolution amplitude is thus written as a multiple integral

$$
\langle q(t), t|q(0), 0\rangle =
$$

(22)
\[
= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \langle q(t) | e^{-i\frac{\hbar}{\epsilon}H_{\epsilon}} | q_{n-1} \rangle dq_{n-1} \cdots dq_{1} \langle q_{1} | e^{-i\frac{\hbar}{\epsilon}H_{\epsilon}} | q(0) \rangle,
\]

(23)

where \( q_l = q(l\epsilon) \) and \( H \) is the Hamilton operator. Each factor—the short-time propagator—is then identified with an exponential of the short-time action involving an approximation of the classical Lagrangian,

\[
\langle q_{l+1} | e^{-i\frac{\hbar}{\epsilon}H_{\epsilon}} | q_{l} \rangle = \frac{1}{A} e^{i\frac{\epsilon}{2\hbar}L(q_{l+1},q_{l})\epsilon},
\]

(24)

with normalization factor \( A \). For instance, for a non-relativistic particle of mass \( m \)

\[
H = \frac{\hat{p}^2}{2m} + V(\hat{q}),
\]

(25)

and one computes (via momentum representation)

\[
\frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} \exp\left( \frac{i}{\hbar} \left( p_{l}q_{l+1} - \frac{p_{l}^2}{2m} - V(q_{l}) \right) \epsilon \right) dp_{l} =
\]

(26)
\[
= (\frac{2\pi i \hbar \varepsilon}{m})^{-\frac{1}{2}} \exp\left(\frac{i}{\hbar} \left(\frac{1}{2} m \left(\frac{q_{l+1} - q_l}{\varepsilon}\right)^2 - V(q_l)\right)\right),
\]

i.e.

\[
L(q_{l+1}, q_l) = \frac{1}{2} m \left(\frac{q_{l+1} - q_l}{\varepsilon}\right)^2 - V(q_l) \quad \text{and} \quad A = \left(\frac{2\pi i \hbar \varepsilon}{m}\right)^{\frac{1}{2}}.
\]

A sequence of \(q_l\)'s for each \(t_l\) shall, in the limit, define a path of the system and each of the integrals is to be taken over the entire range available to each \(q_l\). In other words, the multiple integral is taken over all possible paths.

Let us return to our analogue of a free non-relativistic particle. The above approach will guide us in our \(N \times N\) approximation with the short-time propagator induced by the unitary operator \(C_{N1}\). In this approximation \(q_l \approx a\eta_N \rho_l\), so, for a single time step, \(\langle q_{l+1}, \varepsilon|q_l, 0\rangle\) is approximated by

\[
\langle q_{l+1}, \varepsilon|q_l, 0\rangle a\eta_N = \langle q_{l+1}|e^{-\frac{i}{\hbar} H\varepsilon}|q_l\rangle a\eta_N =
\]
\[ = \langle \rho_{l+1} | C_{N1} | \rho_l \rangle = \frac{1}{N} \sum_{j_l=0}^{N-1} \omega_N^{-\frac{1}{2}j_l^2} + (\rho_{l+1} - \rho_l)j_l. \]

For \( \tau \) time steps

\[ \langle \rho_{\tau}, \tau \varepsilon | \rho_0, 0 \rangle = \sum_{\rho_1, \ldots, \rho_{\tau-1}} \langle \rho_{\tau} | C_{N1} | \rho_{\tau-1} \rangle \cdots \langle 1 | C_{N1} | 0 \rangle = (29) \]

\[ = \langle \rho_{\tau} | C_{N1}^{\tau} | \rho_0 \rangle = \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{-\frac{1}{2}j^2_{\tau} + (\rho_{\tau} - \rho_0)j}. \] (30)

The above Gauss-like sum for a single time-step can be summed up using C.L. Siegel’s Reciprocity Formula for generalized Gauss sums

\[ \sum_{n=0}^{|c|-1} e^{\pi i(an^2 + bn)/c} = \sqrt{|c|} \sum_{n=0}^{|a|-1} e^{-\pi i(cn^2 + bn)/a} \] (31)
valid for $a, b, c \in \mathbb{Z}$, $ac \neq 0$, $ac + b$ even. Putting $a = N$ with $N$ odd, $c = 1$, $n = j_l$ and $b = -2\rho - 1$ with $\rho = \rho_{l+1} - \rho_l$ one obtains

$$\frac{1}{N} \sum_{j_l=0}^{N-1} \frac{1}{\omega_N} j_l(j_l-1) + (\rho_{l+1} - \rho_l) j_l = \frac{1}{\sqrt{iN}} \omega_N^\frac{1}{2} (\rho_{l+1} - \rho_l + \frac{1}{2})^2.$$  \hspace{1cm} (32)

On the basis of this formula we prefer the operator

$$C_{N2} |j\rangle = \omega_N^{-\frac{1}{2}} j(j-1) |j\rangle,$$  \hspace{1cm} (33)

for unitary single-step time evolution. It satisfies simpler relations than $C_{N1}$,

$$C_{N2}^{-1} Q_N C_{N2} = Q_N P_N, \quad C_{N2}^{-1} P_N C_{N2} = P_N,$$  \hspace{1cm} (34)

$$C_{N2}^{-s} Q_N^\rho P_N^j C_{N2}^s = Q_N^\rho P_N^{\rho s}.$$  \hspace{1cm} (35)

With operator $C_{N2}$ we compute in position representation

$$\langle \rho_{l+1} | C_{N2} | \rho_l \rangle = \frac{1}{\sqrt{iN}} \omega_N^\frac{1}{2} (\rho_{l+1} - \rho_l + \frac{1}{2})^2.$$  \hspace{1cm} (36)
This result can be interpreted as the emergence of a dimensionless Lagrangian $\mathcal{L}_N$,

$$
\langle \rho_{l+1} | C_{N2} | \rho_l \rangle = \frac{1}{\sqrt{iN}} \mathcal{L}_N(\rho_{l+1}, \rho_l), \quad \mathcal{L}_N(\rho_{l+1}, \rho_l) = \frac{1}{2} (\rho_{l+1} - \rho_l + \frac{1}{2})^2.
$$

(37)

In order to go over to the $1 + 1$ space-time and obtain the corresponding local Lagrangian $L_N$, we divide by $a\eta_N$ and express the short-time propagator

$$
\langle q_{l+1}, \varepsilon | q_l, 0 \rangle = \frac{1}{a\eta_N} \langle \rho_{l+1} | C_{N2} | \rho_l \rangle =
$$

$$
\frac{1}{a\eta_N} \frac{1}{\sqrt{iN}} \omega_N^{\frac{1}{2}} (\rho_{l+1} - \rho_l + \frac{1}{2})^2 = \left( \frac{2\pi i \hbar \varepsilon}{m} \right)^{-\frac{1}{2}} e^{\frac{i}{\hbar} \frac{1}{2} m (\frac{q_{l+1} - q_l + \frac{a\eta_N}{\varepsilon}}{\varepsilon})^2} 2\varepsilon.
$$

(38)

(39)

This result can be rewritten

$$
\frac{1}{\sqrt{iN}} \omega_N^{\mathcal{L}_N(\rho_{l+1}, \rho_l)} = \left( \frac{2\pi i \hbar \varepsilon}{m} \right)^{-\frac{1}{2}} e^{\frac{i}{\hbar} L_N(q_{l+1}, q_l) \varepsilon a\eta_N},
$$

(40)
where the phase factor appearing in the short-time propagator is seen to define the corresponding small increment of the action $L_N \varepsilon$ which is proportional to the local Lagrangian

$$L_N = \frac{1}{2} m \left( \frac{ql_1 - ql + \frac{\alpha \eta_N}{2}}{\varepsilon} \right)^2.$$  \hfill (41)

Note that in the limit $N \rightarrow \infty$, we have $\eta_N = \sqrt{2\pi/N} \rightarrow 0$ and obtain the usual form of the short-time propagator for the free quantum particle.

The potential field $V(q)$ can be also incorporated in the short-time propagator.
Short-time propagator and mutually unbiased bases

Let us denote the bases composed of eigenvectors of the operators $Q_N^j P_N^\sigma$ by $\mathcal{B}_{(j,\sigma)}$. Unitary operator $C_{N2}$ (or $C_{N1}$) plays analogous role as operator $D_N$ in our previous study of mutually unbiased bases for prime $N$. There the iterations of $D_N$ generated the maximal set of $N + 1$ mutually unbiased bases

$$\mathcal{B}_{(1,0)} \xrightarrow{S_N} \mathcal{B}_{(0,1)} \xrightarrow{D_N} \mathcal{B}_{(1,1)} \xrightarrow{D_N} \mathcal{B}_{(2,1)} \xrightarrow{D_N} \cdots \xrightarrow{D_N} \mathcal{B}_{(N-1,1)}$$

starting with the canonical basis $\mathcal{B}_{(1,0)}$.

If $N$ is prime, identical reasoning as in shows that the iterations of unitary operator $C_N$ (indices 1 or 2 omitted) generate in a similar way another maximal set of $N + 1$ mutually unbiased bases

$$\mathcal{B}_{(0,1)} \xrightarrow{S_N^{-1}} \mathcal{B}_{(1,0)} \xrightarrow{C_N} \mathcal{B}_{(1,1)} \xrightarrow{C_N} \mathcal{B}_{(2,1)} \xrightarrow{C_N} \cdots \xrightarrow{C_N} \mathcal{B}_{(1,N-1)},$$
now starting with the momentum basis $B_{(0,1)}$. Thus the composite unitary operators $C_N^b S_N^{-1}$, $b = 0, 1, \ldots, N - 1$ produce all the bases of the maximal set when applied to the momentum basis.

$N$ is not restricted to primes but may take arbitrary odd value. Notwithstanding this general situation the formulae derived in previous section show that the bases appearing in the short-time propagators, i.e. $\{|\rho\rangle\} = B_{(1,0)}$ and $\{C_N \omega_N |\rho\rangle\} = B_{(1,1)}$ are mutually unbiased. Especially, for the short-time propagator

$$|\langle\rho_{l+1}|C_N \omega_N |\rho_l\rangle| = \frac{1}{\sqrt{N}}$$

holds and it is this constant absolute value that entails mutual unbiasedness of the bases involved. From the physical viewpoint the state evolving after a short time interval $\varepsilon$ carries no information about the preceding state. The trivial information-theoretic meaning of mutual unbiasedness therefore consists in the fact that in each single evolution step complete loss of information
occurs. Physical information is carried only by the phase factor whose phase is proportional to the local Lagrangian.

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