From associative algebroids to associative algebras

Perhaps it is enough to say

assoc. algebras : assoc. algebroids = groups : groupoids.

By definition, an assoc. algebroid is a vector bundle \( A \to M \times M \), together with maps \( A_{(x,y)} \otimes A_{(y,z)} \to A_{(x,z)} \) depending smoothly on \( x, y, z \in M \) and associative in the obvious sense (one can also use a Lie groupoid in place of \( M \times M \)).

There is an associative product on \( \Gamma(A \otimes \det T^*|^{1/2}(M \times M)) \) defined by

\[
\alpha \ast \beta(x, z) = \int_{y \in M} \alpha(x, y) \cdot \beta(y, z)
\]

(we should make some restriction on behaviour of \( \alpha \) and \( \beta \) at infinity to make the integral convergent). This formula defines associative algebra structure on the algebra \( D(A) \) of \( A \)-differential operators (pseudodif. oper. with support (not just singular support) in the diagonal).

Clifford algebroids, symbol calculus, and nilpotent Diracs

Let \( V \) be a vector space with inner product. One constructs \( \text{Spin}(V) \subset \text{Cl}(V) \) in the usual way. There is a natural \( \mathbb{Z}/(2) \)-graded involution \( t : \text{Cl}(V) \otimes \mathbb{C} \to \text{Cl}(V) \otimes \mathbb{C} \) (i.e. \( t^2 = \text{the parity operator}, \lambda^t = \lambda \) for \( \lambda \in \mathbb{C} \), and \( (ab)^t = (-1)^{|a||b|}ab^t \)) defined uniquely by \( v^t = iv \) for \( v \in V \). Clearly \( g^t = g^{-1} \) for \( g \in \text{Spin}(V) \). \( \text{Cl}(V) \) is filtered, with the associated graded Poisson algebra \( \bigwedge^* V \); there (i.e. if we pass to symbols) \( t \) becomes \( \text{degree} \).

As a generalization, let \( A \to M \) be a vector bundle with inner product; we shall define its Clifford algebroid \( \text{Cl}(A) \) on \( M \times M \). Let \( V \) be a vector space with inner product, isomorphic to the fibres of \( A \), and let \( P \) be the corresponding principal \( \text{SO}(V) \)-bundle (i.e. the points of \( P \) are isomorphisms of \( V \) with the fibres of \( A \)). Suppose that \( P \) admits a lift to principal \( \text{Spin}(V) \)-bundle \( \tilde{P} \). Then

\[
\text{Cl}(A) = \tilde{P} \times \tilde{P} \times \text{Cl}(V) / \text{Spin}(V) \times \text{Spin}(V),
\]

where the \( \text{Spin}(V) \times \text{Spin}(V) \)-action is \( (p_1, p_2, a)(g_1, g_2) = (p_1 g_1, p_2 g_2, g_1^{-1} a g_2) \) (recall \( \text{Spin}(V) \subset \text{Cl}(V) \), so that \( g_1^{-1} a g_2 \) makes sense). Multiplication in \( \text{Cl}(A) \) is given by \( (p_1, p_2, a)(p_2, p_3, b) = (p_1, p_3, ab) \) (easily seen to be well defined).

When restricted to the diagonal of \( M \times M \), \( \text{Cl}(A) \) becomes the bundle of Clifford algebras of \( A \). Although \( \text{Cl}(A) \) depends on \( \tilde{P} \), it can be defined over a neighbourhood of the diagonal without use of \( \tilde{P} \); simply replace \( \tilde{P} \times \tilde{P} \) in its definition by a neighbourhood of diagonal of \( P \times P \) and put similar condition on \( \text{Spin}(V) \times \text{Spin}(V) \). Certainly, \( \mathcal{D}(\text{Cl}(A)) \) does not need any spin structure.

We define a map \( t : \text{Cl}(A) \otimes \mathbb{C} \to \text{Cl}(A) \otimes \mathbb{C} \) covering the flip map of \( M \times M \) by \( (p_1, p_2, a)^t = (p_2, p_1, a^t) \). It generates a \( \mathbb{Z}/(2) \)-graded involution on the algebra of \( \text{Cl}(A) \)-valued generalized half-densities and its subalgebras (provided we choose the condition at infinity in a flip-invariant way).

Finally, there is a reasonable symbol calculus on \( \mathcal{D}(\text{Cl}(A)) \), i.e. there is a filtration on \( \mathcal{D}(\text{Cl}(A)) \) such that the associated graded Poisson algebra is graded-commutative. Moreover, the map \( t \) can be used to define subprincipal symbols. Namely, using trivialization and coordinates to make it a differential operator on \( \mathbb{R}^n \) with coefficients in \( \text{Cl}(V) \), the order of a \( \text{Cl}(A) \)-diff. operator is the order of the coefficient (in \( \text{Cl}(V) \)) plus twice the order of the derivative; this definition is clearly independent of the choices.
These symbols are functions on the graded symplectic supermanifold \((E, \omega)\) with \(\deg \omega = 2\), constructed by Alan. I recall its construction. Take \(T^*P \times IV^*\) with \(T^*P\) graded by even numbers and \(IV\) with the usual grading (the standard notation is \(T^*[2]P \times V^*[1]\)); \(E\) is its symplectic reduction at zero. Using a trivialization, \(E\) is locally \(T^*M \times IV^*\); this can be used to define the symbol as the symbol of the coefficient times the symbol of the derivative; again (quite clearly), this is independent of the choices.

When we pass to symbols, \(t\) becomes \(t^\text{degree}\). An operator \(D\) of degree \(k\) will be called selfadjoint if \(D^t = i^k D\). Any \(D\) of degree \(k\) and parity \(k \mod 2\) is uniquely split as \(D = D_1 + D_2\), where \(D_{1,2}\) are selfadjoint and \(\deg D_1 = k, \deg D_2 = k - 2\) (namely \(D_1 = (D + i^{-k}D^t)/2, D_2 = (D - i^{-k}D^t)/2\)). The symbol of \(D_2\) is the subprincipal symbol of \(D\).

Here is an application to Courant algebroids and nilpotent Dirac operators. Suppose \(\theta\) is a cubic function on \(E\) satisfying \(\{\theta, \theta\} = 0\), i.e. we are given a Courant algebroid structure on \(A\). Let \(D\) be the selfadjoint 3rd degree operator with symbol \(\theta\) (because \(D\) is of degree 3, the selfadjointness condition specifies it uniquely). Since \(\{\theta, \theta\} = 0, \{D, D\}\) is of degree at most 2. We have \([D, D]^t = -[D^t, D] = [D, D]\). Hence \([D, D]\) is of degree 0, i.e. \(D^2 = \text{function}\). In other words, for any Courant algebroid, there is a canonical choice for generating nilpotent Dirac operator.

**Symbols via tangent groupoid**

The aim of this section is to keep finite number of dimensions to the very last moment. We shall need assoc. algebroids over groupoids (not just over \(M \times M\)); this generalization is clear (just half-densities on \(M \times M\) are replaced by half densities on \(\alpha\)-fibres times half-densities on \(\beta\)-fibres). We’ll describe an algebroid \(\mathcal{C}(A)\) over the tangent groupoid \(\tau M\); the symbol calculus described above is contained in \(\mathcal{C}(A)\).

I first recall the structure of \(\tau M\). Its base is \(M \times \mathbb{R}\); points in this \(\mathbb{R}\) will be denoted as \(\epsilon\). Morphisms of \(\tau M\) never change \(\epsilon\), i.e. \(\tau M\) can be viewed as a 1-parameter family of groupoids. Over \(\epsilon \neq 0\) the groupoid is just \(M \times M\), while over 0 it is \(TM\). There is an \(\mathbb{R}^*\)-action on \(\tau M\) (scale transformations): for \(\epsilon \neq 0, \epsilon\) is just mapped to \(\epsilon/\lambda\), while over \(\epsilon = 0, TM\) is multiplied by \(\lambda (\lambda \in \mathbb{R}^*)\). There is a natural manifold structure on \(\tau M\) for which this action is smooth.

Now we can describe \(\mathcal{C}(A) \rightarrow \tau M\). Over \(\epsilon \neq 0\) it is \(\mathcal{C}(A)\); over 0, it is a new algebroid \(Gr(A) \rightarrow TM\):

\[
Gr(A) = (TP \times \bigwedge V)/TSO(V).
\]

Here \(TSO(V)\) (the tangent bundle of \(SO(V)\)) is a Lie group in the usual way (it is a semidirect product of \(SO(V)\) with its Lie algebra). Its action on \(TP\) is clear (vectors are added). We identify \(\mathfrak{so}(V)\) with \(\bigwedge^2 V\); \(\epsilon \in \bigwedge^2 V\) acts on \(\bigwedge V\) by multiplication by \(\epsilon^t\), and \(SO(V)\) in the usual way. Composition in \(Gr(A)\) is given by \((e_1, c_1)(e_2, c_2) = (e_1 + e_2, c_1c_2)\), for \(e_{1,2} \in T_xP, c_{1,2} \in \bigwedge V\).

We let \(\mathbb{R}^*\) act on \(\mathcal{C}(A)\): \(\lambda \in \mathbb{R}^*\) acts by \(\lambda^2\) on \(\tau M\); for \(\epsilon \neq 0\) it leaves \(\mathcal{C}(A)\) intact and for \(\epsilon = 0\) it multiplies \(V\) (or \(A\)) by \(\lambda\). Again, there is a natural smooth structure on \(\mathcal{C}(A)\) for which this action is smooth.

Finally, the convolution algebra of \(Gr(A)\) is canonically isomorphic to the algebra of functions on the supermanifold \(E\): choosing a trivialization of \(V\) (a 1st order trivialization, i.e. a connection, is sufficient), this isomorphism is simply the Fourier transform along the fibres of \(TM\); obviously (and miraculously as well), it is independent of the choice.

The symbol calculus is ready now. Given an element of the convolution algebra of \(\mathcal{C}(A)\), its restriction to \(Gr(A)\) is its symbol. A \(\mathcal{C}(A)\)-pseudodiff. operator of order \(k\) is uniquely extended to a weight-\(k\) \(\mathbb{R}^*\)-equivariant \(\mathcal{C}(A)\)-operator.