Introduction to Quantum Electrodynamics

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These are lecture notes devoted to introductory chapters of Quantum Electrodynamics (QED). The notes consist of two chapters:

1. The Dirac field and the relativistic invariance
- The Lorentz transformations and relativistic fields
- The Dirac equation and its solutions, polarization sums
- Dirac field quantization, field energy and momentum, charge
- Fermions, the Dirac field propagator

2. Quantum electrodynamics and Feynman rules
- QED equations of motion, Gauss law, Coulomb gauge
- Free transversal electromagnetic field and its quantization
- The interaction picture and the perturbation theory
- Self-interacting scalar field, Feynman rules
- QED in Coulomb gauge, gauge invariance
- The relativistic formalism and Feynman rules

This two chapters should be followed by a part devoted to simple applications of Feynman perturbation technique:

3. Elementary processes in QED
- Scattering amplitudes and the differential cross-section
- Kinematics of binary processes, decay rate of an unstable particle
- The scattering $e^- e^+ \rightarrow \mu^- \mu^+$, the square of the scattering amplitude
- Unpolarized scattering and its differential cross-section
- The scattering $e^- \mu^- \rightarrow e^- \mu^-$, the square of the scattering amplitude
- Crossing symmetry, Mandelstam variables, crossed channels
- Compton scattering $e^- \gamma \rightarrow e^- \gamma$

- The polarization sum for photons - Ward identity
- Klein-Nishina formula for the cross-section of unpolarized $e^- \gamma$ scattering
- The annihilation $e^- e^- \to \gamma \gamma$, crossing symmetry and cross-section

1 The Dirac field and its relativistic invariance

1.1. Lorentz transformations

We shall label the points of the Minkowski space-time as follows:

$$x = (x^\mu) = (x^0, x^1, x^2, x^3) = (t, \vec{x}), \tag{1.1}$$

where $t = x^0$ denotes time and $\vec{x} = (x^1, x^2, x^3)$ labels the space position. The scalar product of two 4-vectors $x = (x^\mu)$ a $y = (y^\mu)$ in Minkowski space-time is given by

$$x.y = x^\mu \, \eta_{\mu\nu} \, y^\nu = x^\mu \, y_\mu \,, \tag{1.2}$$

where

$$y_\mu = \eta_{\mu\nu} \, y^\nu \,, \quad y_\mu = \eta_{\mu\nu} \, y^\nu \,.$$

We lower or rise the indices with the help of the relativistic metric tensor $(\eta_{\mu\nu})$ =diag$(1, -1, -1, -1)$ or its inverse $(\eta^{\mu\nu})$ =diag$(1, -1, -1, -1)$):

$$\eta_{\mu\nu} \, \eta^{\nu\rho} = \eta_\mu^\rho = \delta_\mu^\rho \,,$$

where δ_μ^ρ is the Kronecker symbol defined by: $\delta_\mu^\rho = 1$ for $\mu = \rho$ and $\delta_\mu^\rho = 0$ for $\mu \neq \rho$. We adopt the *Einstein summation convention*: we sum over the same repeated upper and lower indices, e.g. $x_\nu\, y^\nu \;=\; x_0\, y^0 \,+\, x_1\, y^1 \,+\, x_2\, y^2 \,+\, x_3\, y^3$.

Let us consider the linear transformation which preserves the relativistic scalar product $x.y \;=\; x^\mu\, \eta_{\mu\nu}\, y^\nu$ of any two 4-vectors x and x:

$$x^\mu \;\mapsto\; \Lambda^\mu{}_\nu\, x^\nu\,, \quad y^\mu \;\mapsto\; \Lambda^\mu{}_\nu\, y^\nu\,. \tag{1.3}$$

Let us rewrite the scalar product in matrix notation $x.y \;=\; x^T\, \eta\, y$, where y denotes the column with 4 components y^μ, x^T is a row with 4 components x^μ and η is 4×4 matrix with elements $\eta_{\mu\nu}$. Similarly the transformation law in matrix notation can be written as follows: $x \;\mapsto\; x' \;=\; \Lambda\, x$ a $y \;\mapsto\; y' \;=\; \Lambda\, y$ where Λ is the 4×4 matrix with elements $\Lambda^\mu{}_\nu$. The invariance of the scalar products induces a constraint on admissible matrices Λ:

$$x^T\, \eta\, y \;=\; x'^T\, \eta\, y' \;=\; x^T\, \Lambda^T\, \eta\, \Lambda\, y \quad\Leftrightarrow\quad \Lambda^T\, \eta\, \Lambda \;=\; \eta\,. \tag{1.4}$$

Here, Λ^T is the transposed matrix of the matrix Λ. Such matrices form a Lie group, called *Lorentz group*. The elements of the Lorentz group which can be expressed in exponential form

$$\Lambda \;=\; \exp\left(-i\omega J\right) \quad\Leftrightarrow\quad \Lambda^\alpha{}_\beta \;=\; \left(\exp\left(-i\omega J\right)\right)^\alpha{}_\beta \tag{1.5}$$

The symbol ω in the exponent is a real number and J is 4×4 matrix satisfying condition

$$J^T\, \eta \;+\; \eta\, J \;=\; 0 \quad\Leftrightarrow\quad J^T \;=\; -\eta\, J\, \eta\,. \tag{1.6}$$

This condition is a direct consequence of (1.4) and (1.5). There are 6 independent 4×4 matrices $J_{\mu\nu} \;=\; J_{\nu\mu}$ satisfying (1.6). Their $\alpha\beta$ matrix elements are given as:

$$\left(J^{\mu\nu}\right)_{\alpha\beta} \;=\; i\, \left(\eta^\mu_\alpha\, \eta^\nu_\beta \;-\; \eta^\nu_\alpha\, \eta^\mu_\beta\right)\,. \tag{1.7}$$

In this form we can freely rise and lower all indices simultaneously on both sides.

In any proper Lorentz transformation $\Lambda \;=\; \exp\left(-iJ\right)$ the exponent J is given as a linear combination

$$J \;=\; \frac{1}{2}\, \omega_{\mu\nu}\, J^{\mu\nu} \quad \mu\, \nu \;=\; 0, 1, 2, 3\,, \tag{1.8}$$

specified by 6 real parameters $\omega_{\mu\nu} \;=\; -\omega_{\nu\mu}$. The matrix $J^{\mu\nu}$ generate Lorentz transformations in $(\mu\nu)$-plane in Minkowski space:

• Three generators J_ij, $i, j = 1, 2, 3$, generate rotations in 3-space (for its specification we need 3 parameters - 3 Euler angles θ, ϕ, ψ);

• Three generators J_0j, $j = 1, 2, 3$, generate boosts (the transformation to a system moving with a speed $\vec{v}$ with respect to the original reference frame - this requires again 3 parameters).

For infinitesimal Lorentz transformation, specified by infinitesimal parameters $\omega_{\mu\nu}$ we obtain

$$x_\alpha \;\mapsto\; \left(\exp\left(-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\right)^{\alpha}_{\;\;\beta} x^\beta$$

$$= \left(\delta^\alpha_\beta \;-\; \frac{i}{2}(J^{\mu\nu})^\alpha{}_\beta \;+\; \ldots\right) x^\beta \;=\; x^\alpha \;-\; \frac{i}{2}(J^{\mu\nu})^\alpha{}_\beta x^\beta \;+\; \ldots$$

$$x_\alpha \;\mapsto\; x_\alpha \;+\; \omega_{\alpha\beta}\, x^\beta\,, \quad \omega_{\alpha\beta} \;=\; -\omega_{\alpha\beta}\,.$$

In the last step we have used the explicit formula (1.7).

It can be easily shown that the matrices $J_{\mu\nu}$ satisfy the commutation relations for Lorentz group generators, i.e. defining relations of Lie algebra $so(3,1)$:

$$[J^{\mu\nu}, J^{\rho\sigma}] \;=\; -\,i\,(\eta_{\nu\rho}J^{\mu\sigma} \;-\; \eta_{\mu\rho}J^{\nu\sigma} \;+\; \eta_{\mu\sigma}J^{\nu\rho} \;-\; \eta_{\nu\sigma}J^{\mu\rho})\,. \qquad (1.9)$$

Finally, we point out that it holds

$$J^{\mu\nu}\, x^\rho \;=\; i\,(\eta^{\mu\rho}\, x^\nu \;-\; \eta^{\nu\rho}\, x^\mu)\,. \qquad (1.10)$$

This formula is equivalent to the relation

$$x^\mu \;\mapsto\; \Lambda^\mu{}_\nu\, x^\nu\,, \quad \Lambda \;=\; \exp\left(-\frac{i}{2}\omega_{\mu\nu}\, J^{\mu\nu}\right) \qquad (1.11)$$

which tell us that 4 real numbers $x = (x^\mu)$, $\mu = 0, 1, 2, 3$,transform as relativistic 4-vector.

1.2. Relativistic scalar fields. The relativistic scalar field $\phi(x)$ is described by a (real or complex) function defined in all points of Minkowski space-time: $x \mapsto \phi(x)$. By definition, under Lorentz transformation $x \mapsto \Lambda\, x$ the field $\phi(x)$ is transforming in the following way:

$$\phi(x) \;\mapsto\; T(\Lambda)\phi(x) \;=\; \phi(\Lambda^{-1}x)\,. \qquad (1.12)$$

In (1.12) Λ^{-1} denotes the inverse matrix of the matrix Λ. The symbol $T(\Lambda)$ represents the linear operator defined by the last equation. The assignment $\Lambda \mapsto T(\Lambda)$ defines the Lorentz group representation because it "copies" the group product:

$$T(\Lambda_1)\, T(\Lambda_2) \;=\; T(\Lambda_1 \Lambda_2)\,, \quad T(\mathbf{1}) \;=\; \mathbf{I}\,.$$

The symbol $\mathbf{1}$ denotes the 4×4 unit matrix (corresponding to the unity in group) and the symbol I is the unit operator corresponding to the identity map: $\phi(x) \mapsto \phi(x)$.

Under infinitesimal Lorentz transformation $x_\alpha \mapsto x_\alpha + \omega_{\alpha\beta}\, x^\beta$ the field transforms as follows

$$\begin{aligned} \phi(x) \;\mapsto\; \phi(\Lambda^{-1}x) \;&=\; \phi(x_\alpha \,-\, \omega_{\alpha\beta}\, x^\beta) \\ &=\; \phi(x) \;-\; \frac{i}{2}\omega_{\mu\nu}\, (\mathcal{J}^{\mu\nu}\phi)(x)\,. \end{aligned}$$

Comparing the last expression with the Taylor expansion of the field on a first line, we obtain the formula for the generator of Lorentz transformations $\mathcal{J}^{\mu\nu}$ which acts on fields as a 1-st order differential operator:

$$\mathcal{J}^{\mu\nu} \;=\; -i\, (x^\mu\, \partial^\nu \,-\, x^\nu\, \partial^\mu)\,, \quad \partial^\nu \;=\; \eta^{\mu\nu}\, \partial_\nu\,, \tag{1.13}$$

where $\partial_\nu \;=\; \partial_{x^\nu}$. It can be easily shown that the differential operators $\mathcal{J}^{\mu\nu} \;=\; -\mathcal{J}^{\nu\mu}$ again satisfy the commutation relations (1.9) for Lorentz group generators.

Multi-component relativistic fields. Let us consider n-component field

$$\psi(x) \;=\; \begin{pmatrix} \psi_1(x) \\ \dots \\ \psi_n(x) \end{pmatrix}$$

with components $\psi_a(x)$, $a \;=\; 1, \dots, n$. We shall assume that under Lorentz transformation $x \;\to\; \Lambda\, x$ the field components transform as follows:

$$\psi_a(x) \;\mapsto\; S_a{}^b(\Lambda)\, \psi_b(\Lambda^{-1}x) \;\equiv\; (T(\Lambda)\, \psi(x))_a\,. \tag{1.14}$$

The mapping $\psi(x) \;\mapsto\; T(\Lambda)\, \psi(x)$ will generate the Lorentz group representation:

$$T(\Lambda_1)\, T(\Lambda_2) \;=\; T(\Lambda_1\, \Lambda_2)\,, \quad T(\mathbf{1}) \;=\; I$$

exactly, when $\Lambda \mapsto S_a^b(\Lambda)$ will be the $(n \times n)$-matrix representation of the Lorentz group

$$S_a{}^b(\Lambda_1)\, S_b{}^c(\Lambda_2) \;=\; S_a{}^c(\Lambda_1\,\Lambda_2)\,, \quad S_a{}^b(\mathbf{1}) \;=\; \delta_a^b\,.$$

Example: As an important example of multi-component field can serve the *relativistic vector field* $V^\mu(x)$ which under Lorentz transformations maps as follows:

$$V^\mu(x) \;\mapsto\; \Lambda^\mu{}_\nu\, V^\nu(\Lambda^{-1}x)\,. \tag{1.15}$$

We leave as an exercise to derive, in this case, the form of the differential operators $\mathcal{J}^{\mu\nu}$ generating Lorentz transformations.

1.3. Particles and relativistic fields. In the framework of quantum theory, the relativistic particle with mass m and 4-momentum $p = (E_{\vec{p}}, \vec{p})$, i.e., with the 3-momentum $\vec{p}$ and energy $E_{\vec{p}}$, is described by the de Broglie wave function

$$\frac{1}{(2\pi)^{3/2}}\, e^{-ip.x} \;=\; \frac{1}{(2\pi)^{3/2}}\, e^{-\,iE_pt\,+\,i\vec{p}.\vec{x}}\,, \quad E_p \;\equiv\; E_{\vec{p}} \;=\; \sqrt{\vec{p}^2 \,+\, m^2}\,. \tag{1.16}$$

Below we shall frequently use the notation E_p instead of $E_{\vec{p}}$ (a similar simplification we shall frequently use for some other quantities too).

The set of such particles with different momenta, which do not possess other internal degrees of freedom, is described by relativistic scalar field

$$\phi(x) \;=\; \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\sqrt{2E_p}}\, \left(a_p\, e^{-ip.x} \;+\; b_p^*\, e^{+ip.x}\right)\,, \quad p \;=\; (E_p,\, \vec{p})\,, \tag{1.17}$$

i.e. the scalar field is represented by a complex linear combination of de Broglie wave functions and complex conjugated wave functions (the numerical factor in front of the integral and the factor $\sqrt{2E_p}$ in the measure represent just a convenient normalization):

• The first integrand describes a system of free particles with 3-momenta $\vec{p}$, the complex coefficient $a_p \equiv a_{\vec{p}}$ is proportional to the probability amplitude that the a-particle with 3-momentum $\vec{p}$ is contained in the ensemble of particles;

• The second integrand is a linear combination of complex conjugated wave functions of b-particles with 3-momenta $\vec{p}$, the coefficient $b_p \equiv b_{\vec{p}}$ is proportional to the probability amplitude that the b-particle with 3-momentum $\vec{p}$ is contained in the ensemble of particles.

Note: For *complex* scalar fields the coefficients a_p a b_p are independent (there is no relation among them). Simply, we have two sorts of particles: a-particles and b-particles which are *antiparticles* to a-particles. Particles and antiparticles have the same mass but they possess opposite electric charge (and all other charges they possess are opposite too).

The reality condition $\phi(x) = \phi^*(x)$ for *real* scalar field implies constraint $a_p = b_p^*$. The system contains one kind of particles: the particle a is identical to its antiparticle, the particles possess zero charges.

The free scalar field is a solution of *Klein-Gordon equation*

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = 0 . \tag{1.18}$$

The relativistic invariance of Klein-Gordon equation. Under Lorentz transformation $x \mapsto \Lambda x$ the scalar field $\phi(x)$ $\phi(x) \mapsto \phi_\Lambda(x) = \phi(\Lambda^{-1}x)$, kde

$$\phi(\Lambda^{-1}x) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\sqrt{2E_p}} \left(a_p \, e^{-ip.\Lambda^{-1}x} + b_p^* \, e^{+ip.\Lambda^{-1}x} \right) .$$

Taking into account, that under substitution $p' = \Lambda p$

$$\frac{d^3\vec{p}}{2E_p} = \frac{d^3\vec{p}\,'}{2E_{p'}} , \tag{1.19}$$

and that $p.\Lambda^{-1}x = p^T \eta \, \Lambda^{-1} x = p^T \Lambda^T \eta \, x = (\Lambda p)^T \eta \, x$, we can write

$$\phi_\Lambda(x) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}\,'}{\sqrt{2E_{p'}}} \left(a'_{p'} \, e^{-ip'.x} + b'^*_{p'} \, e^{+ip'.x} \right) , \tag{1.20}$$

where

$$a'_{p'} = \sqrt{\frac{E_{p'}}{E_p}} \, a_p , \quad b'_{p'} = \sqrt{\frac{E_{p'}}{E'_p}} \, b_p , \quad p' = \Lambda p . \tag{1.21}$$

We see that the transformed field $\phi_\Lambda(x)$ is again a solution of the Klein-Gordon equation (1.18):

$$(\partial_\mu \partial^\mu + m^2)\phi_\Lambda(x) = 0,$$

however, with expansion coefficients transformed according to (1.21).

We have constructed a representation of the Lorentz group realized in space of field configurations. In fact, we have two independent unitary representations: the first one in the space of particle configurations and the other one in the space of antiparticle configurations (the coefficients a_p a b_p are independent, and the unitarity is due to the positivity of the integral measure).

1.4. The Dirac equation

The free particle relativistic equation, the Klein-Gordon equation, had been first suggested by E. Schrödinger. He used the quantization rule: the energy E and 3-momentum $\vec{p}$ should be replaced in the Hamiltonian (formula for the energy) by differential operators

$$E \mapsto i\partial_t, \quad p_j \mapsto i\partial_j, \tag{1.22}$$

where $\partial_j = \partial_{x^j}$, $j = 1, 2, 3$. However, he found problems with relativistic formulation of the hydrogen atom problem. Therefore, he applied the rule (1.22) within the non-relativistic formula for the electron energy moving in the Coulomb field of proton. He solved and published his well-known Schrödinger equation.

The relativistic version of the quantum equation of motion was published after by Klein and Gordon (the equation was known to V. A. Fock too). The fact, that due to relativistic invariance the Klein-Gordon equation contained *second order* time derivative (beside second order position derivatives) leads to various complications with quantum-mechanical interpretation of the formalism.

This fundamental problems motivated P. A. M. Dirac to search for a relativistic equation describing the free particle with mass m containing just the *first order* space-time derivatives. Dirac postulated the equation in the form

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0, \quad \partial_\mu = \partial_{x^\mu}, \tag{1.23}$$

with the coefficients γ^μ being unknown constant objects.

Dirac determined γ's from the requirement that (1.23) should contain particle-like solutions, i.e., the solutions of the first order Dirac equation (1.23) should simultaneously satisfy the second order Klein-Gordon equation.

Multiplying (1.23) with the operator $(i\gamma^\nu \partial_\nu + m)$ we obtain second order equation

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0\,,$$
$$(\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m^2)\,\psi(x) = 0\,. \tag{1.24}$$

Since the differential operator ∂_μ commutes with any other ∂_ν we can write

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\,\partial_\nu \partial_\mu\,.$$

Equation (1.24) will be consistent with Klein-Gordon equation (1.18) provided the coefficients γ^μ satisfy *anticommutation* relations:

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\eta^{\mu\nu}\,. \tag{1.25}$$

This follows directly from

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\,\partial_\nu \partial_\mu = \eta^{\nu\mu}\,\partial_\nu \partial_\mu = \partial^\mu \partial_\mu\,.$$

Equations (1.25) represent the well-known defining relations for the (real) Clifford algebra generators:

• pre $\mu \neq \nu$ the generators anti-commute: $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu$, and

• they are normalized by: $(\gamma^0)^2 = +1$ and $(\gamma^i)^2 = -1$, $i = 1, 2, 3$.

The algebra of γ-matrices (1.25) may be realized in terms of 4×4 complex matrices, so called Dirac matrices. In the Weyl (chiral) basis they are given in 2×2 block form by the formulas:

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} \mathbf{0} & \sigma^j \\ -\sigma^j & \mathbf{0} \end{pmatrix}, \quad j = 1, 2, 3\,. \tag{1.26}$$

All entering elements are 2×2 matrices: $\mathbf{0}$ is the zero matrix, $\mathbf{1}$ is the unit matrix and σ^j, $j = 1, 2, 3$, are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.27}$$

Note: Dirac used a different realization of γ-matrices:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} \mathbf{0} & \sigma^j \\ -\sigma^j & \mathbf{0} \end{pmatrix}, \quad j = 1,2,3. \tag{1.28}$$

This Dirac (or standard) realization of γ-matrices is equivalent to the Weyl realization. In what follows we shall use the Weylovu representation.

Products and linear combinations of matrices γ^μ generate the algebra of all 4×4 matrices. The conventional basis of this matrix algebra is formed by the following 16 matrices:

$$\begin{aligned} \mathbf{1}, \ \gamma^\mu, \ S^{\mu\nu} &= \frac{i}{4}[\gamma^\mu, \gamma^\nu] \\ \tilde{\gamma}^\mu &= \gamma^5 \gamma^\mu, \ \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 . \end{aligned} \tag{1.29}$$

The indices μ, ν take the values 0,1,2,3:

• here the symbol $\mathbf{1}$ represents the 4×4 unit matrix (we do not introduce a new notation - from the context it will be clear the size of the unit matrix in question),

• 4 matrices γ^μ have been introduced earlier in (1.26) in Weyl representations (or (1.28) in Dirac representation);

• since $S^{\mu\nu} = -S^{\nu\mu}$ we have 6 independent $S^{\mu\nu}$ matrices;

• we add 4 matrices $\tilde{\gamma}^\mu$ and the matrix $\tilde{\gamma}^4 \equiv -i\gamma^5$.

In Weyl basis those matrices have the form

$$S^{0j} = -\frac{i}{2}\begin{pmatrix} \sigma^j & \mathbf{0} \\ \mathbf{0} & -\sigma^j \end{pmatrix}, \quad S^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & \mathbf{0} \\ \mathbf{0} & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\varepsilon^{ijk}\,\Sigma^k ,$$

$$\tilde{\gamma}^0 = -\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \tilde{\gamma^j} = -\begin{pmatrix} \mathbf{0} & \sigma^j \\ \sigma^j & \mathbf{0} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{0} & \mathbf{1} \end{pmatrix}. \tag{1.30}$$

All indices $i, j, k, \ldots$, take values $1, 2, 3$ (the summation convention is understood). These matrices are chosen so that all expressions $\bar{\psi}(x)\, A\, \psi(x)$ are real for A being some of the 16 matrices $\mathbf{1}$, γ^μ, $S^{\mu\nu}$, $\tilde{\gamma}^\mu$ and $i\gamma^5$ (in fact, they form the Lie algebra basis of the conformal group $SO(4,2)$).

Exercise: Find the form of matrices $S^{\mu\nu}$, $\tilde{\gamma}^\mu$ and $i\gamma^5$ in Dirac realization.

The properties of γ-matrices.

• Hermitian conjugation

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 \quad \Leftrightarrow \quad \gamma^{0\dagger} = \gamma^0 , \;\; \gamma^{j\dagger} = \gamma^j , \;\; j = 1,2,3 .$$

• The properties of γ^5

$$(\gamma^5)^2 = \mathbf{1} , \quad \gamma^{5\dagger} = \gamma^5 , \quad \{\gamma^5, \gamma^\mu\} = 0 , \;\; [\gamma^5, S^{\mu\nu}] = 0 .$$

Spinor representations of Lorentz group.

It can be easily shown, using the anti-commutation relations for Dirac γ^μ matrices, that the matrices $S^{\mu\nu}$, $\mu, \nu = 0,1,2,3$, satisfy the commutation relations for Lorentz group Lie algebra:

$$[S^{\mu\nu}, S^{\rho\sigma}] = -i\,(\eta_{\nu\rho} S^{\mu\sigma} - \eta_{\mu\rho} S^{\nu\sigma} + \eta_{\mu\sigma} S^{\nu\rho} - \eta_{\nu\sigma} S^{\mu\rho}) . \tag{1.31}$$

We see that we have obtained a 4-dimensional representation of the Lorentz group in the space $\mathbf{C}^4$. However, it is a reducible representation, since the generators $S^{\mu\nu}$ are 2×2-block diagonal:

• the 2×2 matrices in the left upper corner form a 2-dimensional spinor representation of the Lorentz group (the fundamental representation of the group $SL(2, \mathbf{C})$ of 2×2 matrices with the unit determinant),

• similarly, the 2×2 matrices in the right lower corner form a 2-dimensional conjugated spinor representation of the Lorentz group (the anti-fundamental representation of the group $SL(2, \mathbf{C})$).

The linear combinations of matrices $S^{\mu\nu}$ (and their exponents) can act, besides the space $\mathbf{C}^4$, in the space of matrices. If A is a 4×4 matrix, then $S^{\mu\nu}$ act as commutators (so called, adjoint action or adjoint representation):

$$A \mapsto [S^{\mu\nu}, A] .$$

In particular, we have

$$[S^{\mu\nu}, \mathbf{1}] = 0 , \quad [S^{\mu\nu}, \gamma^5] = 0 . \tag{1.32}$$

That means that the unit matrix **1** and the matrix $i\gamma^5$ transform as scalars under adjoint action by $S^{\mu\nu}$.

Similarly, we have

$$[S^{\mu\nu}, \gamma^\rho] \; = \; (\eta_{\mu\rho}\, \gamma^\nu \; - \; \eta_{\nu\rho}\, \gamma^\mu)\,,$$
$$[S^{\mu\nu}, \tilde{\gamma}^\rho] \; = \; (\eta_{\mu\rho}\, \tilde{\gamma}^\nu \; - \; \eta_{\nu\rho}\, \tilde{\gamma}^\mu)\,. \tag{1.33}$$

These relations tell us that quartets γ^μ and $\tilde{\gamma}^\mu$) transform as 4-vectors under adjoint action by $S^{\mu\nu}$.

Finally, the commutation relations (1.31) for $S^{\mu\nu}$ express the fact that the set of matrices $S^{\mu\nu}$, $\mu\,\nu \; = \; 1, 2, 3$, transforms as an (anti-symmetric) tensor.

Transformation properties of the Dirac (spinor) field.

We postulate that under Lorentz transformation Λ the Dirac field $\psi(x)$ transforms as follows:

$$\psi(x) \;\mapsto\; S(\Lambda)\, \psi(\Lambda^{-1} x)\,,$$
$$S(\Lambda) \; = \; S(\exp(-\frac{i}{2}\omega_{\mu\nu}\, S^{\mu\nu}))\,, \quad \text{pre} \;\; \Lambda \; = \; \exp(-\frac{i}{2}\omega_{\mu\nu}\, J^{\mu\nu})\,. \tag{1.34}$$

As follows from (1.14), this is a representation of Lorentz group in the space of spinor fields $\psi(x)$.

Besides $\psi(x)$, it is convenient to introduce the Dirac conjugated spinor field

$$\bar{\psi}(x) \; = \; \psi(x)^\dagger\, \gamma^0 \; = \; (\psi_3^*(x),\, \psi(x)_4^*,\, \psi_1^*(x),\, \psi(x)_2^*) \tag{1.35}$$

(the last formula is valid in the Weyl representations of γ-matrices). If $\psi(x)$ is a solution of Dirac equation the Dirac conjugated spinor field satisfies the conjugated Dirac equation:

$$i(\partial_\mu\, \bar{\psi})(x)\, \gamma^\mu \; = \; -\, m\, \bar{\psi}(x)\,. \tag{1.36}$$

The follows directly from the z Dirac equation for $\psi(x)$ rewritten in the form

$$i\gamma^\mu\, (\partial_\mu\, \psi)(x) \; = \; m\, \psi(x)\,. \tag{1.37}$$

By hermitian conjugation we obtain

$$-i\partial_\mu \psi^\dagger(x)\, \gamma^{\mu\dagger} \;=\; m\, \psi^\dagger(x)\,.$$

Multiplying this equation by γ^0 from the right and using the relations

$$(\gamma^0)^2 \;=\; \mathbf{1}\,, \quad \gamma^0\, \gamma^\mu\, \gamma^0 \;=\; \gamma^{\mu\dagger}\,, \quad \psi^\dagger(x)\, \gamma^0 \;=\; \bar{\psi}(x)\,,$$

we arrive directly at the conjugated Dirac equation (1.36).

The conjugated Dirac field transforms under Lorentz transformations as follows:

$$\bar{\psi}(x) \;=\; \psi(x)^\dagger\, \gamma^0 \;\mapsto\; \psi(\Lambda^{-1}x)^\dagger\, S^\dagger(\Lambda)\, \gamma^0 \;=\; \bar{\psi}(\Lambda^{-1}x)\gamma^0\, S^\dagger(\Lambda)\, \gamma^0\,.$$

Expressing $S(\Lambda) \;=\; S(\exp\,(-\frac{i}{2}\omega_{\mu\nu}\, S^{\mu\nu}))$ and using

$$\gamma^0\, S^{\mu\nu\,\dagger}\, \gamma^0 \;=\; -\, S^{\mu\nu}$$

we obtain $\gamma^0\, S^\dagger(\Lambda)\, \gamma^0 \;=\; S^{-1}(\Lambda)$. This formula gives the desired transformation for the conjugated Dirac spinor field:

$$\bar{\psi}(x) \;\mapsto\; \bar{\psi}(\Lambda^{-1}x)\, S^{-1}(\Lambda)\,. \qquad (1.38)$$

Transformations of bilinear expressions. The transformation rules for spinor and conjugated spinor fields

$$\psi(x) \;\mapsto\; S(\Lambda)\, \psi(\Lambda^{-1}x)\,, \quad \bar{\psi}(x) \;\mapsto\; \bar{\psi}(\Lambda^{-1}x)\, S^{-1}(\Lambda)\,, \qquad (1.39)$$

allow to derive the transformation rules for bilinear expressions:

$$\begin{array}{rcl}
S(x) \;=\; \bar{\psi}(x)\, \psi(x) & \mapsto & \bar{\psi}(\Lambda^{-1}x)\, \psi(\Lambda^{-1}x) \;=\; S(\Lambda^{-1}x) \\
P(x) \;=\; \bar{\psi}(x)\, i\gamma^5\, \psi(x) & \mapsto & \bar{\psi}(\Lambda^{-1}x)\, i\gamma^5\, \psi(\Lambda^{-1}x) \;=\; P(\Lambda^{-1}x) \\
V^\mu(x) \;=\; \bar{\psi}(x)\, \gamma^\mu\, \psi(x) & \mapsto & \Lambda^\mu{}_\nu\, \bar{\psi}(\Lambda^{-1}x), \gamma^\nu\, \psi(\Lambda^{-1}x) \;=\; \Lambda^\mu{}_\nu\, V^\nu(\Lambda^{-1}x) \\
A^\mu(x) \;=\; \bar{\psi}(x)\, \gamma^5\gamma^\mu\, \psi(x) & \mapsto & \Lambda^\mu{}_\nu\, \bar{\psi}(\Lambda^{-1}x)\, \gamma^5\gamma^\nu\, \psi(\Lambda^{-1}x) \;=\; \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}x) \\
T^{\mu\nu}(x) = \bar{\psi}(x) S^{\mu\nu} \psi(x) & \mapsto & \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \bar{\psi}(\Lambda^{-1}x) S^{\rho\sigma} \psi(\Lambda^{-1}x) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}(\Lambda^{-1}x)
\end{array}$$

1. The transformation rules for $S(x)$ a $P(x)$ tell us that the corresponding bilinear expressions transform as scalar fields. The first expression is a direct

consequence of transformation rules (1.39), in derivation of the second one relation $\gamma^5 \, S(\Lambda) \; = \; S(\Lambda)\, \gamma^5$ is needed.

2. The transformation rules for $V^\mu(x)$ a $A^\mu(x)$ mean that the corresponding bilinear expressions transform as vector fields. Besides the transformation rules 1.39), we need the relation

$$S^{-1}(\Lambda)\, \gamma^\mu \, S(\Lambda) \; = \; \Lambda^\mu{}_\nu \, \gamma^\nu \, , \tag{1.40}$$

for the derivation of the rule for $V^\mu(x)$, in the derivation of the rule for $A^\mu(x)$ we have to take into account the fact that $S(\Lambda)$ commutes with γ^5.

3. The transformation rules for $T^{\mu\nu}(x)$ tell us that $T^{\mu\nu}(x)$ transforms as a (antisymmetric) 2-nd order tensor. In the derivation we need the explicit formula (1.29) for $S^\mu \nu$ and the relation (1.40).

Appendix A: The Fierz identities

Relativistic invariance of the Dirac equations.

We shall show, that the field

$$\psi_\Lambda(x) \; = \; S(\Lambda)\, \psi(\Lambda^{-1} x)$$

is a solution of the Dirac equation

$$(i\gamma^\mu \, \partial_\mu \, - \, m)\, \psi_\Lambda(x) \; = \; 0 \, .$$

Using (1.40) we can put the left-hand side into the form

$$(i\gamma^\mu \, \partial_\mu \, - \, m)\, S(\Lambda)\, \psi(\Lambda^{-1}x) \; = \; S(\Lambda)\, (i\Lambda^\mu{}_\nu \, \gamma^\nu \, \partial_\nu \, - \, m)\, \psi(\Lambda^{-1}x) \, .$$

Introducing the new variable $x' \; = \; \Lambda^{-1} x$ we can express the partial derivatives as follows

$$\partial'_\nu \; \equiv \; \partial_{x'^\nu} \; = \; (\Lambda^{-1})_\nu{}^\mu \, \partial_\mu \; = \; \Lambda^\mu{}_\nu \, \partial_\mu \, .$$

This equation is a direct consequence of the relation $\eta\, \Lambda \; = \Lambda^{-1T} \eta$. We see that the Dirac equation for $\psi_\Lambda(x)$ is equivalent to

$$S(\Lambda)\, (i\gamma^\nu \, \partial'_\nu \, - \, m)\, \psi(x') \; = \; 0 \, .$$

The last equality follows from the fact that $\psi(x)$ is a solution of Dirac equation.

Particle solutions of Dirac equation.

We search the solution of Dirac equation which is proportional to the de Broglie wave function describing free particle with mass m, momentum $\vec{p}$ and energy $E_p = \sqrt{\vec{p}^2 + m^2} > 0$. Such solution is proportional to the plane wave

$$\psi(x) \sim u(p)\, e^{-ip.x}\,, \quad p = (E_p, \vec{p})\,.$$

Inserting such $\psi(x)$ into Dirac equation and the formula $i\partial_\mu\, e^{-ip.x} = p_\mu\, e^{-ip.x}$ we obtain for the spinor $u(p)$ the algebraic equation:

$$(\gamma^\mu\, p_\mu - m)\, u(p) = 0\,. \tag{1.41}$$

Since our γ-matrices possess 2×2 block form we rewrite the 4-component spinor $u(p)$ in terms of two 2-component spinors $u_L(p)$ and $u_R(p)$

$$u(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix}.$$

Taking now γ^μ in Weyl representation we obtain the system of two entangled algebraic equations (we omit the argument p in $u_{L,R}(p)$ and simply E instead of E_p):

$$\begin{pmatrix} -m & E-\vec{p}.\vec{\sigma} \\ E+\vec{p}.\vec{\sigma} & -m \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{array}{l} (E-\vec{p}.\vec{\sigma})\, u_R = m\, u_L \\ (E+\vec{p}.\vec{\sigma})\, u_L = m\, u_R \end{array}. \tag{1.42}$$

With the help of formulas

$$(E-\vec{p}.\vec{\sigma})\,(E+\vec{p}.\vec{\sigma}) = (E+\vec{p}.\vec{\sigma})\,(E-\vec{p}.\vec{\sigma}) = E^2 - \vec{p}^2 = m^2$$

it can be easily seen that the solution of (**??**) is given as

$$u_L(p) = \sqrt{E-\vec{p}.\vec{\sigma}}\,\xi\,, \quad u_R(p) = \sqrt{E+\vec{p}.\vec{\sigma}}\,\xi\,, \tag{1.43}$$

where ξ is an arbitrary *constant* 2-component spinor. The square roots $\sqrt{E\mp\vec{p}.\vec{\sigma}}$ should be understood as Taylor expansions of

$$\sqrt{E\mp\vec{p}.\vec{\sigma}} = \sqrt{E}\,\sqrt{1\mp E^{-1}\vec{p}.\vec{\sigma}} = \sqrt{E}\left(1 \mp \frac{1}{2}\frac{\vec{p}.\vec{\sigma}}{E}\ \ldots\right).$$

Since eigenvalues of the matrix $E^{-1}\vec{p}.\vec{\sigma}$ are in absolute value less than 1, the expansion in powers of $(E^{-1}\vec{p}.\vec{\sigma})^n$ is well defined.

There are two linear independent constant spinors ξ, we shall denote them by ξ^s, $s = \pm 1/2$. We choose them orthonormal, so that it holds

$$\xi^{r\dagger}\xi^s = \delta^{rs}, \quad r, s = \pm 1/2. \tag{1.44}$$

In this way we obtain 2 linear independent particle solutions of Dirac equation

$$u^s(p) = \begin{pmatrix} \sqrt{E - \vec{p}.\vec{\sigma}}\,\xi^s \\ \sqrt{E + \vec{p}.\vec{\sigma}}\,\xi^s \end{pmatrix} \quad s = \pm 1/2, \tag{1.45}$$

which satisfy normalization conditions

$$\bar{u}^s(p)\,u^s(p) = 2m\,\delta^{rs}. \tag{1.46}$$

We search the solution proportional to conjugated de Broglie wave functions in the form:

$$\psi(x) = v(p)\,e^{+ip.x}, \quad p = (E_{\vec{p}}, \vec{p}).$$

As we shall see, they describe antiparticles with energy $E_{\vec{p}}$ and momentum $\vec{p}$. Performing the similar steps as before, we obtain two linear independent (antiparticle) solutions

$$v^s(p) = \begin{pmatrix} \sqrt{E - \vec{p}.\vec{\sigma}}\,\eta^s \\ -\sqrt{E + \vec{p}.\vec{\sigma}}\,\eta^s \end{pmatrix}, \quad s = \pm 1/2, \tag{1.47}$$

depending on two orthonormal constant spinors η^s, $s = \pm 1/2$ satisfying

$$\eta^{r\dagger}\eta^s = \delta^{rs}, \quad r, s = \pm 1/2. \tag{1.48}$$

We point out that the bases $\{\xi^s\}$ and $\{\eta^s\}$ are not related to each other and we can choose them independently. The solutions $v^s(p)$ are normalized, up to sign, as $u^s(p)$ and antiparticle solutions are orthogonal to particle solutions:

$$\begin{aligned} \bar{v}^s(p)\,v^s(p) &= -2m\,\delta^{rs}, \\ \bar{u}^s(p)\,v^s(p) &= \bar{v}^s(p)\,u^s(p) = 0. \end{aligned} \tag{1.49}$$

The general solution $\psi(x)$ of Dirac equation is given as a linear combination of particle and antiparticle plane wave solutions

$$\psi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(a_p^s\,u^s(p)\,e^{-ipx} + b_p^{s\dagger}\,v^s(p)\,e^{ipx}\right). \tag{1.50}$$

Similarly, can be expressed the Dirac conjugated solution

$$\bar{\psi}(x) \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(b_p^s\,\bar{v}^s(p)\,e^{-ipx} \;+\; a_p^{s\dagger}\,\bar{u}^s(p)\,e^{ipx}\right)\,. \qquad (1.51)$$

Example: Prove the following formulas

$$u^{r\dagger}(\vec{p})\,u^s(\vec{p}) \;=\; v^{r\dagger}(\vec{p})\,v^s(\vec{p}) \;=\; 2\,E_p\,\delta^{rs},$$
$$u^{r\dagger}(\vec{p})\,v^s(-\vec{p}) \;=\; v^{r\dagger}(-\vec{p})\,u^s(\vec{p}) \;=\; 0\,. \qquad (1.52)$$

We have used the notation $u^s(\vec{p}) = u^s(p)$ for $p = (E_{\vec{p}}, \vec{p})$, and similarly $v^s(-\vec{p}) = v^s(p')$ pre $p' = (E_{\vec{p}}, -\vec{p})$. In what follows we shall use such notation whenever it is convenient.

Spin sums.

The spin sums express the completeness of the found solutions of Dirac equation. Our goal is the calculation of the following sums

$$\sum_{s=\pm 1/2} u_a^s(p)\,\bar{u}_b^s(p)\,, \qquad \sum_{s=\pm 1/2} v_a^s(p)\,\bar{v}_b^s(p)\,.$$

The indices $a,\, b = 1, 2, 3, 4$, label the components of spinors $u^s(p)$, $v^s(p)$ and Dirac conjugated spinors $\bar{u}^s(p)$, $\bar{v}^s(p)$. Thus, the searched sums can be interpreted as 4×4 matrices (acting in spinor spaces).

Let us consider the first sum

$$\sum_s u_a^s(p)\,\bar{u}_b^s(p) \;=\; \sum_s \begin{pmatrix} \sqrt{E-\vec{p}.\vec{\sigma}}\,\xi^s \\ \sqrt{E+\vec{p}.\vec{\sigma}}\,\xi^s \end{pmatrix}_a \left(\xi^{s\dagger}\sqrt{E+\vec{p}.\vec{\sigma}},\, \xi^{s\dagger}\sqrt{E-\vec{p}.\vec{\sigma}}\right)_b\,.$$

Taking into account the completeness relation $\sum_s \xi_\alpha^s\,\xi_\beta^{s\dagger} = \delta_{\alpha\beta}$, $\alpha,\, \beta = 1, 2$, valid for the orthonormal basis of 2-component spinor $\{\xi^s\}$, we obtain

$$\sum_s u_a^s(p)\,\bar{u}_b^s(p) \;=\; \begin{pmatrix} \sqrt{E-\vec{p}.\vec{\sigma}}\,\sqrt{E+\vec{p}.\vec{\sigma}} & \sqrt{E-\vec{p}.\vec{\sigma}}\,\sqrt{E-\vec{p}.\vec{\sigma}} \\ \sqrt{E+\vec{p}.\vec{\sigma}}\,\sqrt{E+\vec{p}.\vec{\sigma}} & \sqrt{E+\vec{p}.\vec{\sigma}}\,\sqrt{E-\vec{p}.\vec{\sigma}} \end{pmatrix}_{ab}$$

$$= \begin{pmatrix} m & E-\vec{p}.\vec{\sigma} \\ E+\vec{p}.\vec{\sigma} & m \end{pmatrix}_{ab}\,.$$

Now, taking into account the explicit form of γ-matrices we can rewrite the spin sum as follows:

$$\sum_s u_a^s(p)\,\bar{u}_b^s(p) \;=\; (\gamma^\mu\, p_\mu \,+\, m\,\mathbf{1})_{ab} \;\Leftrightarrow\; \sum_s u^s(p)\,\bar{u}^s(p) \;=\; \gamma^\mu\, p_\mu \,+\, m\,\mathbf{1}\,, \tag{1.53}$$

where $\mathbf{1}$ denotes the 4×4 unit matrix. In the second formula we suppressed the matrix (spin) indices - this for of spin sums is frequently used.

The formula for the second polarization sum can be derived analogously:

$$\sum_s v_a^s(p)\,\bar{v}_b^s(p) \;=\; (\gamma^\mu\, p_\mu \,-\, m\,\mathbf{1})_{ab} \;\Leftrightarrow\; \sum_s u^s(p)\,\bar{u}^s(p) \;=\; \gamma^\mu\, p_\mu \,-\, m\,\mathbf{1}\,. \tag{1.54}$$

Appendix B: The transformation properties of particle solutions.

Under Lorentz transformation $\Lambda \,=\, \exp(-\frac{i}{2}\,\omega_{\mu\nu}\,J^{\mu\nu})$ the Dirac field transforms as follows:

$$\begin{aligned}\psi(x) \;\mapsto\; S(\Lambda)\,\psi(\Lambda^{-1}x) \;&=\; e^{-\frac{i}{2}\,\omega_{\mu\nu}\,S^{\mu\nu}}\,e^{-\frac{i}{2}\,\omega_{\mu\nu}\,\mathcal{J}^{\mu\nu}}\,\psi(x)\\ &=\; e^{-\frac{i}{2}\,\omega_{\mu\nu}\,(\mathcal{J}^{\mu\nu}\,+\,S^{\mu\nu})}\,\psi(x)\,.\end{aligned}$$

Here $\mathcal{J}^{\mu\nu}$ are generators of Lorentz transformations in Minkowski space (see (1.13)) and $S^{\mu\nu}$ denote the generators in spinor representation (see (1.29)).

Let us act on a-particle part of the Dirac equation

$$\psi(x) \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}}\sum_s \left(a_p^s\,u^s(p)\,e^{-ipx} \;+\; \ldots\right)\,. \tag{1.55}$$

with inverse Lorentz transformation: $\psi(x) \;\mapsto\; S(\Lambda^{-1})\,\psi(\Lambda x)$:

- The matrix $S(\Lambda^{-1})$ just mixes various solutions $u^s(p)$, $s \,=\, 1, 2$,

$$S(\Lambda^{-1})\,u^s(p) \;=\; \sum_{s'} D_{s's}(W(\Lambda,p))u^{s'}(p)\,, \tag{1.56}$$

where the summation is over $s' \,=\, 1, 2$, i.e. $(D_{ss'}(W(\Lambda,p)))$ is 2×2 matrix, its dependence on Λ and p is determined below;

• In integral (1.55) we perform substitution $x \mapsto x' = \Lambda x$, similar steps as by scalar field leads to the transformation rule for expansion coefficients a^s_p:

$$a^s_p \;\mapsto\; \sqrt{\frac{E_{\Lambda p}}{E_p}}\, \sum_{s'} a^{s'}_{\Lambda p}\, D_{s's}(W(\Lambda, p))\,. \tag{1.57}$$

For expansion coefficients b^s_p a similar rule can be derived.

Wigner little group. Wigner proposed following method how to determine the matrix $(D_{ss'}(W(\Lambda, p)))$:

• First we introduce standard rest 4-momentum $k_0 = (m, \vec{0})$ for a particle with m. With boost, i.e. Lorentz transformation determined by matrix $L^\mu{}_\nu(p)$ with components:

$$L^0{}_0(p) \;=\; C(p)\,, \quad L^i{}_0(p) \;=\; L^0{}_i(p) \;=\; S(p)\,\hat{p}_i\,,$$

$$L^i{}_0(j) \;=\; \delta^i_j \,-\, \hat{p}_i\,\hat{p}_j \,+\, \hat{p}_i\,\hat{p}_j\,C(p)$$

we transform the particle to the rest-frame in which the particle possesses the 4-momentum $p = (E_p, \vec{p})$. The matrix elements of $L^\mu{}_\nu(p)$ contain quantities defined as follows:

$$S(p) \;=\; \frac{|\vec{p}|}{m}\,, \quad C(p) \;=\; \sqrt{1 \,+\, S^2(p)} \quad \hat{\vec{p}} \;=\; \frac{\vec{p}}{|\vec{p}|}\,.$$

• For any Lorentz transformation Λ we consider a sequence of transformations

$$k_0 \;\mapsto\; W(\Lambda, p)\, k_0 \;\equiv\; L^{-1}(\Lambda p)\, \Lambda\, L(p)\, k_0 \;=\; L^{-1}(\Lambda p)\, \Lambda\, p \;=\; k_0\,.$$

In the first step we used the relation $L(p)\, k_0 = p$, and in the last one the relation $L^{-1}(p')\, p' = k_0$ valid for any p'.

• We see that it holds $W(\Lambda, p)\, k_0 = k_0$, i.e. the transformation $W(\Lambda, p)$ belongs to the stability group of the standard 4-vector $k_0 = (m, \vec{0})$. However, such transformations are just 3-rotations in space: $W(\Lambda, p) \in SO(3)$, where $SO(3)$ denotes the group of spatial rotations.

• Consequently, the $D(W)$ is 2×2 unitary matrix $D^\dagger(W) = D^{-1}(W)$ (because, 3-rotations in spinor space are generated by hermitian matrices

$S_k = \frac{i}{2}\varepsilon_{ijk} S^{ij}$). It is well-known that unitary irreducible $SO(3)$ representation by $N \times N$ matrices corresponds to spin $s = (N-1)/2$. In our case, $N = 2$, i.e. Dirac particles possess spin $s = 1/2$.

1.5. The Dirac field quantization.

Dirac Hamiltonian. The Dirac Lagrangian density, which Euler-Lagrange equations corresponds just to the Dirac equation, can be chosen as

$$\mathcal{L}_D(\bar{\psi}, \psi) \;=\; \bar{\psi}(x)\,(i\gamma^\mu\,\partial_\mu \;-\; m)\,\psi(x)\,.$$

The canonical field-momentum $\pi(x)$ conjugated to the field $\psi(x)$ is obtained by taking the derivative $\dot{\psi} = \partial_0\psi$ of the Lagrangian density in question:

$$\pi \;=\; \frac{\partial \mathcal{L}_D}{\partial \dot{\psi}} \;=\; i\,\bar{\psi}\,\gamma^0 \;=\; i\,\psi^\dagger\,.$$

We see that the canonical field-momentum $\pi(x)$ is not an independent field variable, but is simply proportional to the Dirac conjugated field $\bar{\psi}(x)$. That means that the system contains, rather trivial, field constraints. They do not influence the description of the system, and we skip their discussion (for details see, e.g. [Weinberg]).

The Hamiltonian of the system in question is given by the formula

$$H \;=\; \int d^3\vec{p}\;(\pi\,\dot{\psi} \;-\; \mathcal{L}_D)$$

Simple calculations give the result

$$H_D \;=\; \int d^3\vec{x}\;\bar{\psi}(x)(-i\vec{\gamma}.\vec{\partial} \,+\, m)\psi(x)\,.$$

This expression is called *Dirac Hamiltonian* and we label it with subscript D.

The energy operator. Let us insert into Dirac Hamiltonian the general solutions of the Dirac equation for $\psi(x)$ and $\bar{\psi}(x)$:

$$\psi(x) \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}}\sum_s \left(a_p^s\,u^s(p)\,e^{-ipx} \;+\; b_p^{s\dagger}\,v^s(p)\,e^{ipx}\right)$$

$$\bar{\psi}(x) \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(b_p^s\, \bar{v}^s(p)\, e^{-ipx} \;+\; a_p^{s\dagger}\, \bar{u}^s(p)\, e^{ipx}\right) . \qquad (1.58)$$

After insertion the formula for H_D reads

$$H_D \;=\; \frac{1}{(2\pi)^6}\int \frac{d^3\vec{p} d^3\vec{p}'}{\sqrt{2E_p\, 2E_{p'}}} \int d^3\vec{x} \sum_{s,s'} \left(b_{p'}^{s'}\, \bar{v}^{s'}(p')\, e^{-ip'x} \;+\; a_{p'}^{s'\dagger}\, \bar{u}^{s'}(p')\, e^{ip'x}\right)$$

$$\times \left(a_p^s\, u^s(p)\, e^{-ipx} \;+\; b_p^{s\dagger}\, v^s(p)\, e^{ipx}\right) . \qquad (1.59)$$

Using the well-known formula

$$\int d^3\vec{x}\, e^{\pm i(\vec{p}\pm\vec{p}').\vec{x}} \;=\; (2\pi)^3\, \delta(\vec{p}\pm\vec{p}')$$

we can perform trivially the integration over $d^3\vec{x}$, and then we can integrate directly over $d^3\vec{p}'$.

Multiplying the two integrand factors $(\ldots) \times (\ldots)$ in 1.60 we obtain four terms. Taking into account the formulas 1.52 (see *Example*)

$$u^{s'\dagger}(\vec{p})\, u^s(\vec{p}) \;=\; v^{s'\dagger}(\vec{p})\, v^s(\vec{p}) \;=\; 2\, E_{\vec{p}}\, \delta^{s's}\, , \;\; u^{s'\dagger}(\vec{p})\, v^s(-\vec{p}) \;=\; v^{s'\dagger}(-\vec{p})\, u^s(\vec{p}) \;=\; 0\, ,$$

we obtain two zero contributions and in the two remaining terms we can perform the summation over s'. In this way we obtain a *preliminary* expression for the Dirac field energy

$$H_D \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(a_p^{s\dagger}\, a_p^s \;-\; b_p^s\, b_p^{s\dagger}\right) . \qquad (1.60)$$

Note: Let us remind that for scalar field an analogous calculation gives the following result for the energy of a real scalar field

$$H \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}}\, \frac{1}{2} \left(a_p^\dagger\, a_p \;+\; a_p\, a_p^\dagger\right) . \qquad (1.61)$$

After quantization the coefficients $a_p \;=\; a_{\vec{p}}$ and $a_p^\dagger \;=\; a_{\vec{q}}^\dagger$ are replaced by operators satisfying the canonical commutation relations for the *bosonic* annihilation and creation operators:

$$[a_p, a_q^\dagger] \;=\; (2\pi)^3\, \delta(\vec{p} \,-\, \vec{q})\, ,$$

$$[a_p, a_q] \;=\; [a_p^\dagger, a_q^\dagger] \;=\; 0\,. \qquad (1.62)$$

The annihilation and creation operators act in Fock space $\mathcal{F}$, which is a Hilbert space spanned by normalized n-particle states

$$|p_1, p_2, \ldots, p_n\rangle \;\sim\; a_{p_1}^\dagger\, a_{p_2}^\dagger\, \ldots\, a_{p_n}^\dagger |0\rangle\,, \qquad (1.63)$$

where the symbol $\sim$ means that we do not indicate explicitly the normalization factor on right hand side. The state $|0\rangle$ is a normalized vacuum state in Fock space, $\langle 0|0\rangle \,=\, 1$, which does not contain any particle and is defined by relations

$$a_p\,|0\rangle \;=\; 0\,, \quad \text{for all } p \,=\, (E_p,\, \vec{p})\,. \qquad (1.64)$$

The scalar field energy given by (1.61) is ill-defined (divergent) in the Fock space.

The consistency of QFT formalism requires that all physical field quantities, such as energy, should be well-defined in Fock space. This is guaranteed when they are given in terms of a *normal products* of annihilation and creation operators: in any term containing products of those operators we put all creation operators $a_q^\dagger$ to the left and all annihilation operators a_p to the right:

$$:a_p\, \ldots\, a_q^\dagger\, \ldots\, a_{q'}^\dagger\, \ldots\, a_{p'}: \;=\; a_q^\dagger\, \ldots\, a_{q'}^\dagger\, a_p\, \ldots\, a_{p'},. \qquad (1.65)$$

The normal product is labeled by $:\ldots:$. The right hand side contains first the product of all creation operators followed the product of all annihilation operators entering the original expression. In particular,

$$:a_p\, a_q^\dagger: \,=\, :a_q^\dagger\, a_p: \,=\; a_q^\dagger\, a_p\,.$$

The normal ordered expression energy is well-defined in the Fock space

$$H \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}}\; :\left(a_p^\dagger\, a_p \;+\; a_p\, a_p^\dagger\right): \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}}\; :a_p^\dagger\, a_p:\,. \qquad (1.66)$$

The multi-particle mean values of energy are positive and in vacuum state the mean energy vanish: $\langle 0|\, H\,|0\rangle \,=\, 0$. Without normal ordering the mean values of energy are all divergent.

Such approach can not be applied in the case of Dirac Hamiltonian. Considering in (1.60) bosonic annihilation and creation operators we would obtain:

$$H_D \;=\; \frac{1}{2\pi^3} \int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(a_p^{s\dagger}\, a_p^s \;-\; b_p^{s\dagger}\, b_p^s \right) \,.$$

Although it is a finite expression in Fock space, it is unbounded from below. However, the energy should be bounded below, otherwise the system will be not stable.

Dirac found an unexpected solution, he postulated that the spinor fields plane wave expansion enter *fermionic* annihilation and creation operators satisfying *anti-commutation* relations:

$$\{a_p^r, a_q^{s\dagger}\} \;=\; \{b_p^r, b_q^{s\dagger}\} \;=\; (2\pi)^3\, \delta(\vec{p} - \vec{q})\, \delta^{rs} \,,$$

$$\{a_p^r, a_q^s\} \;=\; \{a_p^{r\dagger}, a_q^{s\dagger}\} \;=\; \{b_p^r, b_q^s\} \;=\; \{b_p^{r\dagger}, b_q^{s\dagger}\} \;=\; 0 \,.$$

$$\{a_p^r, b_q^{s\dagger}\} \;=\; \{b_p^r, a_q^{s\dagger}\} \;=\; \{a_p^r, b_q^s\} \;=\; \{b_p^{r\dagger}, a_q^{s\dagger}\} \;=\; 0 \,. \qquad (1.67)$$

In particular,

$$(a_p^r)^2 \;=\; (a_p^{s\dagger})^2 \;=\; (b_p^r)^2 \;=\; (b_p^{s\dagger})^2 \;=\; 0 \,. \qquad (1.68)$$

Similarly as in bosonic case, the fermionic annihilation and creation operators act in Fock space $\mathcal{F}$ spanned by multi-particle orthonormal states

$$|(p_1, r_1), \ldots, (p_n, r_n); (q_1, s_1), \ldots, (q_m, s_m)\rangle \;\sim\; a_{p_1}^{r_1\dagger} \ldots a_{p_n}^{r_n\dagger}\, b_{q_1}^{s_1\dagger} \ldots b_{q_m}^{s_m\dagger} |0\rangle \,. \qquad (1.69)$$

State $|0\rangle$ is the normalized vacuum state in Fock space defined as

$$a_p^s\, |0\rangle \;=\; b_p^s\, |0\rangle \;=\; 0 \,, \quad \text{pre } s \;=\; \pm 1/2 \text{ a } \textit{vsetky } p \;=\; (E_p, \vec{p}) \,. \qquad (1.70)$$

Because, the squares of fermionic operators vanish the multi-particle states cannot contain 2 identical particles with same spin and momentum - fermionic particles satisfy *Pauli principle*: the particle occupation numbers in (1.69) are 0 or 1.

As a next step, Dirac had modified adequately the normal product:

$$:a_p^r \ldots b_q^{s\dagger} \ldots a_{q'}^{r'\dagger} \ldots b_{p'}^{s'} : \;=\; \pm\, a_{q'}^{r'\dagger} \ldots b_q^{s\dagger}\, a_p^r \ldots bs'_{p'} \,. \qquad (1.71)$$

This definition is similar to bosonic case but there are essential differences:

• The right hand side contains first the product of all creation operators followed by all annihilation operators entering the expression on left hand side,

• *in addition* the definition contains a sign factor $(-1)^n$, where n is the number of neighbor transpositions needed to reshuffle the rights side to its normal order on left hand side.

In particular,

$$: b_q^{r\dagger}\, b_p^s : \; = \; - \; : b_p^s\, b_q^{r\dagger} : \; = \; b_q^{r\dagger}\, b_p^s \, . \qquad (1.72)$$

Using this normal ordering, the energy of a free Dirac field is represented by a sum of two positive terms:

$$\begin{aligned} H_D \; &= \; \frac{1}{(2\pi)^3} \int d^3\vec{p} \sum_s \; : \left(E_p\, a_p^{s\dagger}\, a_p^s \; - \; E_p\, b_p^s\, b_p^{s\dagger} \right) : \\ &= \; \frac{1}{(2\pi)^3} \int d^3\vec{p} \sum_s \left(E_p\, a_p^{s\dagger}\, a_p^s \; + \; E_p\, b_p^{s\dagger}\, b_p^s \right) . \end{aligned} \qquad (1.73)$$

The momentum operator. The quantum momentum operator is defined by the following expression

$$\vec{P} \; = \; \int d^3\vec{x} \;\; : \bar{\psi}(x)(-\vec{\partial})\psi(x) : , \qquad (1.74)$$

where $\bar{\psi}(x)$ a $\psi(x)$ are given as linear combinations (1.60) of particle solutions of Dirac equation. As we are consider quantum field we introduced in (1.74) the normal product from the very beginning. Performing analogical steps as used for energy operator we obtain the following result:

$$\vec{P} \; = \; \frac{1}{(2\pi)^3} \int d^3\vec{p} \sum_s \left(\vec{p}\, a_p^{s\dagger}\, a_p^s \; + \; \vec{p}\, b_p^{s\dagger}\, b_p^s \right) . \qquad (1.75)$$

The conserved electric charge. The Dirac Lagrangian $\mathcal{L}_D$ is invariant under *global gauge transformations* represented by a constant change of the phase of spinor field:

$$\psi(x) \; \mapsto \; e^{i\alpha}\, \psi(x) \, , \quad \bar{\psi}(x) \; \mapsto \; e^{-i\alpha}\, \bar{\psi}(x) \, , \quad \alpha \; - \; \text{realna konstanta} \, .$$

On a classical (non-quantized) level this invariance leads to the continuity equation for the *current density*:

• Let $\psi(x)$ and $\bar{\psi}(x)$ are solutions of the Dirac equations

$$i\gamma^\mu (\partial_\mu \psi)(x) \ = \ m\,\psi(x)\,,$$
$$i(\partial_\mu \bar{\psi})(x)\,\gamma^\mu \ = \ -\,m\,\bar{\psi}(x)\,.$$

Then,

$$\partial_\mu\,(\bar{\psi}(x)\,\gamma^\mu\,\psi(x)) \ = \ (\partial_\mu \bar{\psi})(x)\,\gamma^\mu\,\psi(x) \ + \ \bar{\psi}(x)\,\gamma^\mu\,(\partial_\mu \psi)(x)$$
$$= \ im\,\bar{\psi}(x)\,\psi(x) \ - \ im\,\bar{\psi}(x)\,\psi(x) \ = \ 0\,. \qquad (1.76)$$

• Using (1.76) it follows directly that the current density

$$j^\mu(x) \ = \ e\,\bar{\psi}(x)\,\gamma^\mu\,\psi(x)\,, \qquad (1.77)$$

satisfies *continuity equation*

$$\partial_\mu\,j^\mu(x) \ = \ 0\,. \qquad (1.78)$$

• The total sa *charge of particles* $Q(t = x_0)$ corresponding to the current $j^\mu(x))$

$$Q(t) \ = \ e\int d^3\vec{x}\,j^0(x) \ = \ e\int d^3\vec{x}\,\bar{\psi}(x)\,\gamma^0\,\psi(x)\,, \qquad (1.79)$$

is conserved provided the positions of particles are restricted to a finite domain in the space, i.e. $\psi(x) = 0$ and $\bar{\psi}(x) = 0$ for $|\vec{x}| \geq R$. The parameter e in(1.79) is the charge of Dirac particle, as we shall see the charge of antiparticle is $-e$. Using continuity equation for $j^\mu(x)$ it can be proved easily that the total charge $Q(t)$ is conserved in time:

$$\dot{Q}(t) \ = \ e\int_{|\vec{x}|<R} d^3\vec{x}\,\partial_0\,j^0(x)$$
$$= \ -e\int_{|\vec{x}|<R} d^3\vec{x}\,\partial_i\,j^i(x) \ = \ e\int_{|\vec{x}|=R} dS^i j^i(x) \ = \ 0\,.$$

In the last step we used the Gauss theorem: we integrated the function $\mathrm{div}\vec{j}(t,\vec{x})$ over 3-ball $|\vec{x}| \leq R$ and used the fact that the 3-current vanish on 3-sphere with radius $\vec{x}| = R$.

In the quantum case the field charge is defined by same expression but with normal ordered charge density (we skip variable t):

$$Q \;=\; e \int d^3\vec{x}\; :\bar{\psi}(x)\,\gamma^0\,\psi(x):\,, \tag{1.80}$$

We insert here the $\bar{\psi}(x)$ a $\psi(x)$ particle solutions of Dirac equation with expansion coefficients being corresponding annihilation and creation operators. Performing similar steps as used for energy operator we obtain the result:

$$Q \;=\; \frac{e}{(2\pi)^3} \int d^3\vec{p} \sum_s \left(a_p^{s\dagger}\, a_p^s \;-\; b_p^{s\dagger}\, b_p^s \right)\,. \tag{1.81}$$

In what follows we shall identify the conserved current $j^\mu(x) \;=\; e :\bar{\psi}(x)\,\gamma^\mu\,\psi(x):$ with the electromagnetic current: the a-particles we identify with electrons with charge e and b-particles with positrons with charge $-e$. The quantity Q describes the total charge of the field.

The interpretation of the free Dirac field. The state of the field with n particles (electrons) and m anti-particles (positrons) with given momenta and spins is given by the following vector normalized vector in Fock space:

$$|(p_1,r_1),\ldots,(p_n,r_n);(q_1,s_1),\ldots,(q_m,s_m)\rangle \;\sim\; a_{p_1}^{r_1\dagger}\,\ldots\,a_{p_n}^{r_n\dagger}\,b_{q_1}^{s_1\dagger}\,\ldots\,b_{q_m}^{s_m\dagger}|0\rangle\,. \tag{1.82}$$

We shall show that (1.82) are eigenstates of the field energy operator H_D, the field momentum operator $\vec{P}$ and the field charge operator Q:

$$\begin{aligned} & H_D\,|(p_1,r_1),\ldots,(p_n,r_n);(q_1,s_1),\ldots,(q_m,s_m)\rangle \\ =\; & \left(\sum_{i=1}^{n} E_{p_i} \;+\; \sum_{j=1}^{m} E_{q_j}\right)\,|(p_1,r_1),\ldots,(p_n,r_n);(q_1,s_1),\ldots,(q_m,s_m)\rangle\,, \end{aligned} \tag{1.83}$$

$$\begin{aligned} & \vec{P}\,|(p_1,r_1),\ldots,(p_n,r_n);(q_1,s_1),\ldots,(q_m,s_m)\rangle \\ =\; & \left(\sum_{i=1}^{n} \vec{p}_i \;+\; \sum_{j=1}^{m} \vec{q}_j\right)\;|(p_1,r_1),\ldots,(p_n,r_n);(q_1,s_1),\ldots,(q_m,s_m)\rangle\,, \end{aligned} \tag{1.84}$$

$$Q\,|(p_1,r_1),\ldots,(p_n,r_n);(q_1,s_1),\ldots,(q_m,s_m)\rangle$$

$$= \; e\,(n-\,)\,|(p_1,r_1),\ldots,(p_n,r_n);(q_1,s_1),\ldots,(q_m,s_m)\rangle\,. \tag{1.85}$$

These eigenvalue equations are a direct consequence of formulas

$$[a^{s\dagger}_p a^s_p,\, a^{s'\dagger}_{p'}] \;=\; \delta(\vec p-\vec p\,')\,\delta^{ss'}\,a^{s\dagger}_p\,,\quad [a^{s\dagger}_p a^s_p,\, a^{s'}_{p'}] \;=\; -\delta(\vec p-\vec p\,')\,\delta^{ss'}\,a^{s'\dagger}_{p'}\,,$$

$$[b^{s\dagger}_p b^s_p,\, b^{s'\dagger}_{p'}] \;=\; \delta(\vec p-\vec p\,')\,\delta^{ss'}\,b^{s\dagger}_p\,,\quad [b^{s\dagger}_p b^s_p,\, a^{s'}_{p'}] \;=\; -\delta(\vec p-\vec p\,')\,\delta^{ss'}\,b^{s\dagger}_p\,. \tag{1.86}$$

The first relation can be obtained as follows:

$$[a^{s\dagger}_p a^s_p,\, a^{s'\dagger}_{p'}] \;=\; a^{s\dagger}_p a^s_p,\, a^{s'\dagger}_{p'} \;-\; a^{s'\dagger}_{p'}\, a^{s\dagger}_p a^s_p, \;=$$

$$a^{s\dagger}_p\, a^s_p,\, a^{s'\dagger}_{p'} \;+\; a^{s\dagger}_p\, a^{s'\dagger}_{p'}\, a^s_p, \;=\; a^{s\dagger}_p\,\{a^s_p,\, a^{s'\dagger}_{p'}\} \;=\; a^{s\dagger}_p\,\delta(\vec p-\vec p\,')\,.$$

Last expression is equivalent to the desired formula (we used the definition of the commutator, then we anti-commutate two creation operators, and finally we used the canonical anti-commutation relation among annihilation and creation operators). The remaining relations can be proved similarly.

The operator identity $[A,BC] \;=\; [A,B]C \;+\; B[A,C]$ gives more general formulas

$$[a^{r\dagger}_p a^r_p,\, a^{r_1\dagger}_{p_1}\,\ldots\,a^{r_n\dagger}_{p_n}] \;=\; \sum_{i=1}^{n}\delta(\vec p-\vec p_i)\,\delta^{rr_i}\,a^{r_1\dagger}_{p_1}\,\ldots\,a^{r_n\dagger}_{p_n}\,,$$

$$[b^{s\dagger}_q b^s_q,\, b^{s_1\dagger}_{q_1}\,\ldots\,b^{s_m\dagger}_{q_n}] \;=\; \sum_{j=1}^{m}\delta(\vec q-\vec q_j)\,\delta^{ss_j}\,b^{s_1\dagger}_{q_1}\,\ldots\,b^{s_n\dagger}_{q_m}\,. \tag{1.87}$$

Equations (1.87) lead directly lead to the eigenvalue equations (1.83)-(1.85) for the energy, momentum and charge operators:

• It follows from (1.83) and (1.85) that the total energy and total momentum of the system are conserved (time independent). Further it follows that the total energy and the total momentum are sums of individual energies and momenta of all individual particles in the state in question. Both properties are typical for ensembles of non-interacting particles: there are no binding energies and the individual energies and momenta of particles do not change in time (there no mutual interaction among particles).

• Similarly, the total charge of the system is conserved. Moreover, the total charge is equal to difference of charges shared by particles (electrons) and that shared by anti-particles (positrons): electrons and positrons have the same mass (the relation between E_p a $\vec p$ is the same for both) but the possess opposite electric charge (the electron charge e and the positron charge is $-e$).

Equal-time canonical anti-commutation relations for Dirac field. Our goal is to prove the anti-commutation relations among spinor fields $\psi(X)$ and $\psi^\dagger(y)$ at *equal time* $x^0 = y^0 = t$:

$$\{\psi_a(t,\vec{x}), \psi_b(t,\vec{y})\} = \{\psi_a^\dagger(t,\vec{x}), \psi_b^\dagger(t,\vec{y})\} = 0 .$$
$$\{\psi_a(t,\vec{x}), \psi_b^\dagger(t,\vec{y})\} = \delta(\vec{x}-\vec{y})\,\delta_{ab} . \qquad (1.88)$$

We shall prove the last relations, the first two can be proved along similar lines. We insert the field expansions

$$\psi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(a_p^s\, u^s(p)\, e^{-ipx} + b_p^{s\dagger}\, v^s(p)\, e^{ipx}\right) , \quad x = (t,\vec{x})$$

$$\psi^\dagger(y) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}'}{\sqrt{2E_{p'}}} \sum_{s'} \left(b_{p'}^{s'}\, v^{s'\dagger}(p')\, e^{-ip'x'} + a_{p'}^{s'\dagger}\, u^{s'\dagger}('p)\, e^{ip'y}\right) , \quad y = (t,\vec{y}) ,$$

into the last anti-commutator in (1.88):

$$\{\psi_a(t,\vec{x}), \psi_b^\dagger(t,\vec{y})\} = \frac{1}{(2\pi)^6}\int \frac{d^3\vec{p}d^3\vec{p}'}{\sqrt{2E_p\,2E_{p'}}} \sum_{s,s'}$$

$$\{a_p^s\, u_a^s(p)\, e^{-ipx} + b_p^{s\dagger}\, v_a^s(p)\, e^{ipx},\, b_{p'}^{s'}\, v_b^{s'\dagger}(p')\, e^{-ip'x'} + a_p^{s\dagger}\, u_b^{s'\dagger}(p')\, e^{ip'x'}\}$$

$$= \frac{1}{(2\pi)^6}\int \frac{d^3\vec{p}d^3\vec{p}'}{\sqrt{2E_p\,2E_{p'}}} \sum_{s,s'}$$

$$\left(\{a_p^s, a_{p'}^{s'\dagger}\}\, u_a^s(p)\, u_b^{s'\dagger}(p')\, e^{-ipx+ip'x'} + \{b_p^s, b_{p'}^{s'\dagger}\}\, v_a^s(p)\, v_b^{s'\dagger}(p')\, e^{+ipx-ip'x'}\right)$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2E_p}$$

$$\left(\sum_s u_a^s(p)\, u_b^{s'\dagger}(p')\, e^{i\vec{p}.(\vec{x}-\vec{y})} + \sum_s v_a^s(p)\, v_b^{s'\dagger}(p')\, e^{-i\vec{p}.(\vec{x}-\vec{y})}\right)$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2E_p}$$

$$\Big((E_p\,\gamma^0 \;-\; \vec{\gamma}.\vec{p} + m)\gamma^0\, e^{i\vec{p}.(\vec{x}-\vec{y})} \;+\; (E_p\,\gamma^0 \;-\; \vec{\gamma}.\vec{p} \;-\; m)\gamma^0\, e^{-i\vec{p}.(\vec{x}-\vec{y})}\Big)_{ab}$$

$$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\, e^{i\vec{p}.(\vec{x}-\vec{y})}\,\delta_{ab} \;=\; \delta(\vec{x}-\vec{y})\,\delta_{ab}\,.$$

During calculations we have used the following steps:

• The anti-commutation canonical relations for annihilation and creation operators

$$\{a_p^s, a_{p'}^{s'\dagger}\} \;=\; \{b_p^s, b_{p'}^{s'\dagger}\} \;=\; (2\pi)^3\,\delta(\vec{p}-\vec{q}\,)\,\delta^{ss'}\,,$$

• the sum rules

$$\sum_s u_a^s(p)\,u_b^{s'\dagger}(p') \;=\; (E_p\,\gamma^0 \;-\; \vec{\gamma}.\vec{p} + m)\gamma^0\,,$$

$$\sum_s v_a^s(p)\,v_b^{s'\dagger}(p') \;=\; (E_p\,\gamma^0 \;-\; \vec{\gamma}.\vec{p} - m)\gamma^0\,,$$

• we replaced $\vec{p} \mapsto -\vec{p}$ in the integral containing $\exp(-i\vec{p}.(\vec{x}-\vec{y}))$, and finally we have used the formula

$$\int d^3\vec{p}\,\exp(i\vec{p}.(\vec{x}-\vec{y})) \;=\; (2\pi)^3\,\delta(\vec{x}-\vec{y})\,.$$

The Dirac field propagator. We shall see later that the free field propagators have a key rôle when the mutual interaction among particles is described within the framework of perturbation theory. First we briefly remind the form of the scalar field propagator, and then we derive its form for the free Dirac field.

The scalar field propagator. Let us consider the free real scalar field

$$\phi(x) \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}}\left(a_p\, e^{-ip.x} \;+\; a_p^\dagger\, e^{+ip.x}\right)\,, \quad p \;=\; (E_p,\,\vec{p})\,. \tag{1.89}$$

Its propagator is defined as the vacuum mean value of the T-product of fields:

$$D_F(x-y) \;=\; \langle 0|\,T[\phi(x)\phi(y)]\,|0\rangle\,. \tag{1.90}$$

The symbol $T[\phi(x)\phi(y)]$ denotes the *time-ordered* product of fields defined by:

$$
\begin{aligned}
T[\phi(x)\phi(y)] &= \phi(x)\phi(y) \quad pre\ x_0 > y_0\,, \\
T[\phi(x)\phi(y)] &= \phi(y)\phi(x) \quad pre\ y_0 > x_0\,.
\end{aligned} \tag{1.91}
$$

Let us consider first the case $x_0 > y_0$, then

$$
D_F(x-y) = \langle 0|\, \phi(x)\phi(y)\, |0\rangle\,.
$$

Inserting here the plane wave expansions (1.89) of $\phi(x)$ and $\phi(y)$ we obtain

$$
D_F(x-y) = \frac{1}{(2\pi)^6} \int \frac{d^3\vec{p}d^3\vec{p}'}{\sqrt{2E_p\, 2E_{p'}}} \langle 0|\, a_p\, a^\dagger_{p'}\, |0\rangle\, e^{-ipx + ip'y}\,. \tag{1.92}
$$

Since,

$$
\langle 0|\, a_p\, a_{p'}\, |0\rangle = \langle 0|\, a^\dagger_p\, a^\dagger_{p'}\, |0\rangle = \langle 0|\, a^\dagger_p\, a_{p'}\, |0\rangle = 0\,.
$$

the potential other three terms do not contribute. Now we use in the formula (1.92) for the propagator the canonical anti-commutation relation written in the form

$$
a_p\, a^\dagger_{p'} = (2\pi)^3\, \delta(\vec{p} - \vec{p}') + a^\dagger_{p'}\, a_p\,.
$$

The first term on r.h.s. does not contribute (again, $\langle 0|\, a_p\, a_{p'}\, |0\rangle = 0$), the contribution from the second term is proportional to δ-function. This allows to perform the integration over $d^3\vec{p}'$, and after that we obtain

$$
D_F(x-y) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E_p}\, e^{-ip(x-y)}\,, \quad p = (E_p,\, \vec{p})\,. \tag{1.93}
$$

Using the same steps for $y_0 > x_0$, we obtain

$$
D_F(x-y) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E_p}\, e^{+ip(x-x')}\,, \quad p = (E_p,\, \vec{p})\,. \tag{1.94}
$$

It can be shown that equations (1.93) and (1.94) can be expressed in terms of a one explicitly *relativistic invariant* formula

$$
D_F(x-y) = \frac{i}{(2\pi)^4} \int d^4p\, \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon}\,. \tag{1.95}
$$

Here $\varepsilon > 0$ is a "small" positive number, and at the end of calculations the limit $\varepsilon \to 0+$ is understood.

The explicit relativistic invariance of the final propagator formula has its price. The 3 dimensional integration (1.93) and (1.94) is replaced by the 4 dimensional one: the zeroth component p_0 is no more fixed to E_p and the integration over dp_0 is added. Therefore, in the relativistic formula (1.95) the 4-momentum p is *off the mass shell* (or simply, off-shell): $p^2 \neq m^2$. This is in contrast with the free particle 4-momentum $p = (E_p, \vec{p})$ which is *on the mass shell* (or simply, on-shell): $p^2 = m^2$.

The propagator for the Dirac field. The Dirac field propagator is again defined as the vacuum mean value of the T-product of fields:

$$S^F ab(x-y) \;=\; \langle 0|\, T[\psi_a(x)\bar{\psi}_b(y)]\, |0\rangle \,. \tag{1.96}$$

Here the symbol $T[\psi_a(x)\bar{\psi}_b(y)]$ denotes the time-ordered product of fermionic Dirac fields which is defined similarly as the T-product of bosonic, but with the distinction that transposition of two fermionic fields changes the sign of the product:

$$\begin{aligned} T[\psi_a(x)\bar{\psi}_b(y)] \;&=\; \psi_a(x)\bar{\psi}_b(y) \quad \text{pre } x_0 \,>\, y_0\,, \\ T[\psi_a(x)\bar{\psi}_b(y)] \;&=\; -\,\psi_a(x)\bar{\psi}_b(y) \quad \text{pre } y_0 \,>\, x_0\,. \end{aligned} \tag{1.97}$$

Let us consider the case $x_0 > y_0$. We insert into formula (1.96) for the propagator the Dirac field plane wave expansions:

$$\begin{aligned} \psi(x) \;&=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(a^s_p\, u^s(p)\, e^{-ipx} \;+\; b^{s\dagger}_p\, v^s(p)\, e^{ipx}\right)\,, \\ \bar{\psi}(x) \;&=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(b^s_p\, \bar{v}^s(p)\, e^{-ipx} \;+\; a^{s\dagger}_p\, \bar{u}^s(p)\, e^{ipx}\right)\,. \end{aligned} \tag{1.98}$$

There will be again only one contribution to the vacuum mean value which contains annihilation operator on the left and the creation on the right:

$$\langle 0|\, T[\psi_a(x)\bar{\psi}_b(y)]\, |0\rangle$$

$$=\; \frac{1}{(2\pi)^6}\int \frac{d^3\vec{p}\,d^3\vec{p}'}{\sqrt{2E_p\, 2E_{p'}}}\; \sum_{s,s'} \langle 0|\, a^s_p\, a^{s'\dagger}_{p'}\, |0\rangle\, u^s_a(p)\, \bar{u}^{s'}_b(p')\, e^{-ipx\,+\,ip'x'}\,.$$

In the next step we use the canonical anti-commutation relation

$$a_p^s\, a_{p'}^{s'\dagger} \;=\; (2\pi)^3\, \delta(\vec{p} - \vec{p}')\delta^{ss'} \;-\; a_{p'}^{s'\dagger}\, a_p^s\,.$$

The non-vanishing contribution proportional comes from the first term: the presence of $\delta(\vec{p} - \vec{p}')$ allows a direct integration over $d^3\vec{p}'$ and $\delta^{ss'}$ allows a summation over s'. Performing these steps we obtain

$$\langle 0|\, T[\psi_a(x)\bar{\psi}_b(y)]\, |0\rangle \;=\; \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E_p} \sum_s u_a^s(p)\, \bar{u}_b^s(p)\, e^{-ip(x-x')}\,.$$

Here we recognize the polarization sum, so that we can write

$$\langle 0|\, T[\psi_a(x)\bar{\psi}_b(y)]\, |0\rangle \;=\; \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E_p}\, (\gamma^\mu p_\mu \,+\, m)_{ab}\, e^{-ip(x-y)}$$

$$= \; \left(i\gamma^\mu \partial_\mu^x \,+\, m\right)_{ab} \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E_p}\, e^{-ip(x-y)} \;=\; \left(i\gamma^\mu \partial_\mu^x \,+\, m\right)_{ab} D_F(x-y)\,, \tag{1.99}$$

where $D_F(x - y)$ denotes the expression valid for scalar field propagator for $x_0 \,>\, y_0$ (we have used the simple formula $i\partial_\mu^x \exp(-ip(x - y)) \,=\, p_\mu \exp(-ip(x-y))$).

Similarly, for $y_0 \,>\, x_0$ we obtain

$$\langle 0|\, T[\psi_a(x)\bar{\psi}_b(y)]\, |0\rangle$$

$$\begin{aligned}
&= \; -\frac{1}{(2\pi)^6} \int \frac{d^3\vec{p} d^3\vec{p}'}{\sqrt{2E_p\, 2E_{p'}}} \sum_{s,s'} \langle 0|b_{p'}^{s'\dagger}\; b_p^s\, |0\rangle\, v_a^s(p)\, \bar{v}_b^{s'}(p')\, e^{+ipx\,-\,ip'y} \\
&= \; -\frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E_p} \sum_s u_a^s(p)\, \bar{u}_b^s(p)\, e^{-ip(x-y)} \\
&\quad -\frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E_p}\, (\gamma^\mu p_\mu \,-\, m)_{ab}\, e^{+ip(x-y)}\,.
\end{aligned} \tag{1.100}$$

Using again the rule $i\partial_\mu^x \exp(+ip(x-y)) \,=\, -p_\mu \exp(-ip(x-y))$ we obtain

$$\langle 0|\, T[\psi_a(x)\bar{\psi}_b(y)]\, |0\rangle$$

$$= \; \left(i\gamma^\mu \partial_\mu^x \,+\, m\right)_{ab} \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E_p}\, e^{+ip(x-y)} \;=\; \left(i\gamma^\mu \partial_\mu^x \,+\, m\right)_{ab} D_F(x-y)\,, \tag{1.101}$$

where $D_F(x-y)$ denotes the expression valid for scalar field propagator for $y_0 > x_0$.

We see that in both cases we obtain the identical expression for the Dirac propagator in terms of $D_F(x-x')$. Finally, we can write

$$\begin{aligned} S_F(x-x') &= \left(i\gamma^\mu \partial^x_\mu + m\mathbf{1}\right) D_F(x-x') \\ &= \frac{i}{(2\pi)^4} \int d^4p \, \frac{\gamma^\mu p_\mu + m}{p^2 - m^2 + i\varepsilon} \, e^{-ip(x-x')} \, . \end{aligned} \tag{1.102}$$

In this final formula we suppressed the matrix indices of the Dirac propagator.

2 The electromagnetic field

With start with Lagrangian describing electromagnetic field acting with an external given source. This is important because the system contains constraints besides equations of motion. The external source is essential for the better understanding of the origin of constraints and their relation to equations of motion. Without external source various important features would be simply lost. Moreover, the obtained results allow a natural generalization to the situation when electromagnetic fields interacts with the system of charged particles. In this case, the electromagnetic field and the ensemble of charged particles will mutually interact - the current of charged particles will represents dynamical source (and not a given external source).

2.1. The electromagnetic field Lagrangian.

In quantum theory the electromagnetic field is specified by 4-potential $A_\mu(x)$, $\mu = 0, 1, 2, 3$. The Lagrangian for electromagnetic field $A_\mu(x)$ interacting with an external electromagnetic current density $J^\mu(x)$ is given as:

$$\mathcal{L}(A, J) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e J^\mu A_\nu . \tag{2.103}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$ is the electromagnetic field strength which is related, in standard way, to the electric field strength $\vec{E}$ and magnetic field strength $\vec{B}$:

$$E^i = F^i_0 = \partial_0 A^i - \partial^i A_0 \Leftrightarrow \vec{E} = \nabla A_0 + \dot{\vec{A}},$$

$$B^k = \frac{1}{2}\varepsilon^{ijk} F_{ij} \Leftrightarrow \vec{B} = \nabla \times \vec{A}. \tag{2.104}$$

The indices i, j, k, take values $1, 2, 3$ (the summation convention is understood), the dot over the symbol means the time derivative, i.e., $\dot{\vec{A}} = \partial_t \vec{A}$, the symbol ∇ denotes the gradient which is the vector differential operator $\nabla = (\partial_{x^1}, \partial_{x^2}, \partial_{x^3}) = -(\partial^1, \partial^2, \partial^3)$.

The charge density ρ and the space current density $\vec{J}$ form the 4-vector of the current density: $J^\mu = (\rho, \vec{J})$.

The Maxwell equations

$$\partial_\mu F^{\mu\nu} = -e J^\nu , \tag{2.105}$$

can be obtained as Euler-Lagrange equations following from the Lagrangian $\mathcal{L}(A, J)$. Since the partial derivatives commute $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ and $F^{\mu\nu} = -F^{\nu\mu}$, from Maxwell equation (2.105) we immediately obtain the continuity equation for the electromagnetic current

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0 , \quad \Rightarrow \quad \partial_\nu J^\nu = 0 . \tag{2.106}$$

Thus, the continuity equation for the electromagnetic current is necessary consistency condition with Maxwell equations (2.105). From the continuity equation directly follows the *electric charge conservation* - the term "current conservation" is frequently used instead of "continuity equation". In what follows, we always assume, that the external electromagnetic current is conserved.

We shall now investigate the Maxwell equations in more detail:

• *The Gauss law.* Let us consider first the time component $\nu = 0$ in (2.105). Since the Lagrangian $\mathcal{L}(A, J)$ does not contain $\partial_0 A^0 = \dot{A}_0$, the corresponding equation does not correspond to equation of motion but a constraint - *the Gauss law* in differential form:

$$\nabla.\vec{E} = \triangle A_0 + \nabla.\dot{\vec{A}} = -e\,\rho , \tag{2.107}$$

where $\triangle = \nabla.\nabla$. Using the well-known formula

$$\triangle \frac{1}{|\vec{x} - \vec{y}|} = -4\pi\,\delta(\vec{x}) - \vec{y})$$

we can express from the Gauss law (2.107) the 0th component of electromagnetic potential (under assumption that the charge density vanish for large $|\vec{x}|$):

$$A_0(x) \equiv A_0(t, \vec{x}) = \frac{1}{4\pi} \int d^3\vec{y}\, \frac{e\rho(t, \vec{x}) + \nabla.\dot{\vec{A}}(t, \vec{y})}{|\vec{x} - \vec{y}|} . \tag{2.108}$$

We see that the Gauss law determines the time-component A_0 of the electromagnetic field potential.

The Maxwell equations represent true equations of motion for the space components A_i, $i = 1,2,3$, of the electromagnetic field potential. The Maxwell equations for vector potential $\vec{A}$ reads:

$$\dot{\vec{E}} - \nabla \times \vec{B} = \vec{J}, \tag{2.109}$$

where the electric field strength $\vec{E} = \dot{\vec{A}} + \nabla A_0$ is the conjugated field momentum to $\vec{A}$ and $\vec{B} = \nabla \times \vec{A}$ is the magnetic field strength.

Field energy - Hamiltonian. The electromagnetic field Hamiltonian is related to the corresponding Lagrangian in a usual way

$$H = \int d^3\vec{x}\,[\vec{E}.\dot{\vec{A}} - \mathcal{L}(A,J)]_{\dot{\vec{A}}\to\vec{E}}$$

$$= \int d^3\vec{x}\,[\vec{E}.\dot{\vec{A}} - \frac{1}{2}\vec{E}^2 + \frac{1}{2}(\nabla\times\vec{A})^2 + e\,\rho\,A_0 - e\,\vec{J}.\vec{A}]\,. \tag{2.110}$$

Here the field component A_0 should be replaced by the Gauss law solution (2.108).

Let us decompose the vector potential $\vec{A}$ to the longitudinal and transversal parts:

$$\vec{A} = \nabla\lambda + \vec{A}_\perp\,, \tag{2.111}$$

where $\lambda = \lambda(t,\vec{x})$ is chosen so that $\vec{A}_\perp$ is just the transversal potential: $\nabla.\vec{A}_\perp - 0$, whereas $\nabla\lambda$ represents longitudinal part. Inserting this decomposition of $\vec{A}$ the formula (2.108) for A_0 takes the form:

$$A_0(t,\vec{x}) = -\dot{\lambda}(t,\vec{x}) + \frac{e}{4\pi}\int d^3\vec{y}\,\frac{\rho(t,\vec{x})}{|\vec{x}) - \vec{y}|}\,. \tag{2.112}$$

For the electric field strength we obtain a similar decomposition to the longitudinal part (first two terms) and transversal part (third term):

$$\vec{E} = \nabla A_0 + \nabla\dot{\lambda} + \dot{\vec{A}}_\perp\,, \tag{2.113}$$

Here, A_0 should be replaced by (2.112).

Now, we insert into Hamiltonian H the decompositions of $\vec{A}$ and $\vec{E}$ into the transversal and longitudinal parts (see equations (2.111) and (2.113)):

$$H = \int d^3\vec{x}\,[\frac{1}{2}\dot{\vec{A}}_\perp^2 + \frac{1}{2}(\nabla\times\vec{A}_\perp)^2 - e\,\vec{J}.\vec{A}_\perp$$

$$+ \frac{1}{2}(\nabla A_0 + \nabla\dot{\lambda})^2 - (\nabla A_0 + \nabla\dot{\lambda}).\nabla A_0 + e\rho A_0 - e\vec{J}.\nabla\lambda,, \quad (2.114)$$

• The first line contains just the transversal part of $\vec{A}_\perp$ and its interaction with the space part $\vec{J}$ of the electromagnetic current density.

• Using the formula (2.112) for A_0 we can rewrite the first term on the second line in the following way

$$\frac{1}{2}\int d^3\vec{x}\,(\nabla A_0 + \nabla\dot{\lambda})^2 = \frac{e^2}{8\pi}\int d^3\vec{x}d^3\vec{y}\,\frac{\rho(t,\vec{x})\,\rho(t,\vec{y})}{|\vec{x} - \vec{y}|}\,. \quad (2.115)$$

This contribution just represents the *Coulomb energy* of external charges.

• Due the Gauss law the second term on second line vanish. Still there is the third term proportional to λ. As we shall show below, the gauge invariance allow us choose $\lambda = 0$ i.e., we can choose the *Coulomb gauge* in which the vector potential is transversal:

$$\vec{A} = \vec{A}_\perp \quad \Leftrightarrow \quad \nabla.\vec{A} = 0\,. \quad (2.116)$$

In Coulomb gauge the Hamiltonian has the form

$$H = \int d^3\vec{x}\,[\frac{1}{2}\dot{\vec{A}}_\perp^2 + \frac{1}{2}(\nabla\times\vec{A}_\perp)^2] \quad (2.117)$$

$$- e\int d^3\vec{x}\,\vec{J}.\vec{A}_\perp + \frac{e^2}{8\pi}\int d^3\vec{x}d^3\vec{y}\,\frac{\rho(t,\vec{x})\,\rho(t,\vec{y})}{|\vec{x} - \vec{y}|}\,, \quad (2.118)$$

• The first line contains just the free transversal electromagnetic field $\vec{A}_\perp$ (since, the Hamiltonian is quadratic);

• the second line contain the interaction of transversal field $\vec{A}_\perp$ with external electromagnetic current $\vec{J}$ and the Coulomb energy external charges.

Quantization of the free electromagnetic field - photons

We shall show that the quantized free transversal electromagnetic field $\vec{A}$, $\nabla.\vec{A} = 0$, describes a system of *photons* - noninteracting particles with vanishing mass.

Let us consider free field case, i.e., with vanishing external current $J^\mu = (\rho, \vec{J}) = 0$. In Coulomb gauge the photon field is given as

$$A = (A^0, \vec{A})\,, \quad \nabla.\vec{A} = 0\,. \quad (2.119)$$

Then the Maxwell equations of motion for A^i reduce to the Klein-Gordon equations

$$\partial_\mu F^{\mu i} \;=\; \Box A^i - \partial^i \partial_\mu A^i \;=\; \Box A^i \quad \Box \;=\; \partial_\mu \partial^\mu \,,$$

for a particle with vanishing mass m. We should supplement it by the transversality of the field $\vec{A}$. Thus, the equations of motion for the photon field reads

$$\Box \vec{A} \;=\; 0\,, \quad \nabla.\vec{A} \;=\; 0\,. \tag{2.120}$$

We expand the solution of (2.119) into plane waves

$$\vec{A}(x) \;=\; \frac{1}{(2\pi)^3} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \sum_{\sigma=1,2} [a_\sigma(k)\, \vec{e}_\sigma(k)\, e^{-ikx} \;+\; a^\dagger_\sigma(k)\, \vec{e}^{\,*}_\sigma(k)\, e^{+ikx}]\,, \tag{2.121}$$

with the photon 4-momentum given as $k \;=\; (\omega_k, \vec{k})$, $\omega_k \;=\; |\vec{k}|$ - this expresses the fact that photons possess vanishing mass $m \;=\; 0$. Further, the symbol $\vec{e}_\sigma(k)$, $\sigma', =\; 1, 2$, represents two complex polarization vectors, $\vec{e}^{\,*}_\sigma(k)$, $\sigma', =\; 1, 2$, are conjugated polarization vectors - choose them normalized and perpendicular to the photon 3-momentum $\vec{k}$:

$$\vec{e}_\sigma(k).\vec{e}^{\,*}_\sigma(k') \;=\; \delta_{\sigma\sigma'}\,, \quad \vec{k}.\vec{e}_\sigma(k) \;=\; \vec{k}.\vec{e}^{\,*}_\sigma(k') \;=\; 0\,. \tag{2.122}$$

The last condition guarantees the transversality of the photon field $\vec{A}$.

The Quantization. We replace the expansion coefficients $a_\sigma(k)$ and $a^\dagger_\sigma(k)$ in (2.120) by annihilation operators $a_\sigma(k)$ and creation operator $a^\dagger_\sigma(k)$, respectively. We postulate for them the boson commutation relations:

$$[a_\sigma(k), a^\dagger_{\sigma'}(k')] \;=\; (2\pi)^3\, \delta(\vec{k} - \vec{k},)\, \delta_{\sigma,\sigma'}$$

$$[a_\sigma(k), a_{\sigma'}(k')] \;=\; [a^\dagger_\sigma(k), a^\dagger_{\sigma'}(k')] \;=\; 0\,. \tag{2.123}$$

The annihilation and creation operators act in the Fock space generated by the action of photon creation operators on vacuum:

$$|(k_1, \sigma_1), \ldots\, (k_n, \sigma_n)\rangle \;\sim\; a^\dagger_{\sigma_1}(k_1)\, \ldots\, {}^\dagger_{\sigma_n}(k_n)\, |0\rangle\,. \tag{2.124}$$

The vacuum state $|0\rangle$ is defined in as usual: $a_\sigma(k)\, 0\rangle \;=\; 0$, $\langle 0|0\rangle \;=\; 1$.

Polarization sum. The polarization vectors $\vec{e}_\sigma(k)$ and $\vec{e}^*_\sigma(k)$, $\sigma = 1, 2$, supplemented by vector $\vec{e}_0(k) = \frac{\vec{e}}{|\vec{e}|}$ form an orthonormal base in the 3-dimensional space of 3-momenta $\vec{k}$. Therefore,

$$e^i_0(k)\, e^j_0(k) \; + \sum_{\sigma=1,2} e^i_\sigma(k)\, e^{j*}_\sigma(k) \; = \; \delta^{ij}\,.$$

From this relation we obtain directly the *polarization sum* formula

$$\sum_{\sigma=1,2} e^i_\sigma(k) \; = \; \delta^{ij} \; - \; \frac{k^i k^j}{\vec{k}^2}\,. \tag{2.125}$$

The photon propagator. The photon propagator is defined as the vacuum mean value of the T-product of photon fields (transversal electromagnetic potential):

$$\langle 0|T[A^i_\perp(x)\, A^j_\perp(x')]0\rangle \; = \; D^{ij}_C(x - x')\,. \tag{2.126}$$

• Inserting for $x_0 > x'_0$ into (2.124) the plane wave expansion (2.120) we obtain:

$$D^{ij}_C(x - x') \; = \; \frac{1}{(2\pi)^6} \int \frac{d^3\vec{k}\, d^3\vec{k}'}{\sqrt{2\omega_k\, 2\omega_{k'}}}$$

$$\times \sum_{\sigma\,\sigma'} e^i_\sigma(k)\, e^{*j}_{\sigma'}(k')\, e^{-ikx\,-\,+ik'x'}\, \langle 0|\, a_\sigma(k)\, a^\dagger_{\sigma'}(k')\, |0\rangle$$

$$= \; \frac{1}{(2\pi)^3} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \sum_\sigma e^i_\sigma(k)\, e^{j*}_\sigma(k)\, e^{-ikx(-x')}\,.$$

Here we have used the relation

$$\langle 0|\, a_\sigma(k)\, a^\dagger_{\sigma'}(k')\, |0\rangle \; = \; \delta_{\sigma\sigma'}\, \delta(\vec{k} - \vec{k}')\,,$$

then we integrated over $d^3\vec{k}'$ and sum over σ'. Taking into account the polarization sum (2.124) we obtain, in the case $x_0 > x'_0$, the formula for propagator

$$D^{ij}_C(x - x') \; = \; \frac{1}{(2\pi)^3} \int \frac{d^3\vec{k}}{2\omega_k} \left(\delta^{ij} \; - \; \frac{k^i k^j}{\vec{k}^2}\right) e^{-ikx(-x')}$$

$$= \; \left(\delta^{ij} \; - \; \triangle^{-1}\, \partial^i \partial^j\right) \frac{1}{(2\pi)^3} \int \frac{d^3\vec{k}}{2\omega_k}\, e^{-ikx(-x')}\,. \tag{2.127}$$

• Similarly, for $x'_0 > x_0$, we obtain:

$$D_C^{ij}(x - x') \;=\; \left(\delta^{ij} \;-\; \triangle^{-1}\,\partial^i\partial^j\right)\,\frac{1}{(2\pi)^3}\int\frac{d^3\vec{k}}{2\omega_k}\;e^{-ikx(-x')}\,. \tag{2.128}$$

In both cases the last integral exactly corresponds the expression for the scalar field propagator. Therefore, introducing off-shell 4-momentum $k = (k_0, \vec{k})$ with arbitrary k_0 (instead of on-shell $k = (\omega_k, \vec{k})$) we obtain, irrespective time-ordering, one common formula for the photon propagator:

$$\begin{aligned} D_C^{ij}(x - x') \;&=\; \left(\delta^{ij} \;-\; \triangle^{-1}\,\partial^i\partial^j\right)\,\frac{i}{(2\pi)^4}\int d^4k\,\frac{e^{-ik(x-x')}}{k^2 \,+\, i\varepsilon} \\ &=\; \frac{1}{(2\pi)^4}\int d^4k\,\left(\delta^{ij} \;-\; \frac{k^i k^j}{\vec{k}^2}\right)\,\frac{i}{k^2 \,+\, i\varepsilon}\;e^{-ikx(-x')}\,. \end{aligned} \tag{2.129}$$

2.2. Lagrangian in quantum electrodynamics (QED)

The quantum electrodynamics (QED) usually means a system charged particles interacting with a bunch of photons. For us, the charged particles will be electrons and positrons described by the Dirac fields $\psi(x)$ and $\bar{\psi}(x)$. We shall take the QED Lagrangian in the form

$$\begin{aligned} \mathcal{L}(A, \psi, \bar{\psi}) \;&=\; -\,\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \;+\; i\bar{\psi}\,\gamma^\mu\,\partial_\mu\psi \;-\; m\,\bar{\psi}\psi \\ &\qquad -\; e\,\bar{\psi}\,\gamma^\nu\,\psi\,A_\nu\,, \end{aligned} \tag{2.130}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$ is the electromagnetic field strength.

• The first line is a free Lagrangian describing noninteracting electromagnetic and Dirac fields, whereas

• the second line describes interaction of the electromagnetic field with the current of charged particles

$$J^\nu \;=\; e\,\bar{\psi}\,\gamma^\nu\,\psi\,. \tag{2.131}$$

The Euler-Lagrange equations are:

• The Maxwell equations for electromagnetic field

$$\partial_\mu F^{\mu\nu} \;=\; -\, e\, J^\nu\,, \tag{2.132}$$

with current J^ν given in (2.129), and

• the Dirac equations for the fields ψ and $\bar{\psi}$ interacting with electromagnetic field

$$i\gamma^\mu\,(\partial_\mu\,\psi)(x) \;=\; m\,\psi(x) \;+\; e\,\gamma^\mu\,\psi(x)\,A_m u(x)\;, \tag{2.133}$$

$$i(\partial_\mu\,\bar{\psi})(x)\,\gamma^\mu \;=\; -\,m\,\bar{\psi}(x) \;-\; e\,\bar{\psi}(x)\gamma^\mu\,A_m u(x)\,. \tag{2.134}$$

The continuity $\partial_\mu\, J^\mu(x) \;=\; 0$, which is necessary for the consistency with Maxwell equations (2.130), follows directly from equations (2.131) and (2.132) for fields $\psi(x)$ and $\bar{\psi}(x)$. The continuity equation is equivalent to the fundamental *conservation law of the total electric charge*

$$Q(t) \;=\; e\int d^3\vec{x}\,\bar{\psi}(x)\,\gamma^0\,\psi(x)\,. \tag{2.135}$$

Thus, $Q(t)$ is constant in time $t \;=\; x_0$, i.e. $\dot{Q}(t) \;=\; 0$.

The QED Hamiltonian. The form of the Hamiltonian can be derived along same lines as we did above in the case of electromagnetic field interacting with an external current (generated by charged particles):

$$H \;=\; \int d^3\vec{x}\,[\frac{1}{2}\,\dot{\vec{A}}_\perp^2 \;+\; \frac{1}{2}\,(\nabla\times\vec{A}_\perp)^2] \;+\; H_D$$

$$-\,e\int d^3\vec{x}\,\vec{J}.\vec{A}_\perp \;+\; \frac{e^2}{8\pi}\int d^3\vec{x}d^3\vec{y}\,\frac{\rho(t,\vec{x})\,\rho(t,\vec{y})}{|\vec{x}\;-\;\vec{y}|} \;-\; e\int d^3\vec{x}\,\vec{J}.\nabla\lambda\,. \tag{2.136}$$

Here H_D is the free Dirac field Hamiltonian

$$H_D \;=\; \int d^3\vec{x}\,\bar{\psi}(-i\vec{\gamma}.\nabla \;+\; m)\psi \tag{2.137}$$

and $\vec{J}$ denotes the 3-density of the electromagnetic current of Dirac particles

$$\vec{J}(x) \;=\; e\,\bar{\psi}(x)\,\vec{\gamma}\,\psi(x)\,. \tag{2.138}$$

We can eliminate the last term in Hamiltonian depending on a longitudinal part $\nabla\lambda(x)$ of the magnetic field by changing properly the phase of Dirac field. Putting

$$\psi(x) \;=\; e^{-ie\lambda(x)}\psi'(x)\,, \quad \bar{\psi}(x) \;=\; e^{ie\lambda(x)}\bar{\psi}'(x)\,. \tag{2.139}$$

we modify the free Dirac Hamiltonian

$$H_D \;=\; \int d^3\vec{x}\,\bar{\psi}'(-i\vec{\gamma}.\nabla \,+\, m)\psi' \;+\; e\int d^3\vec{x}\,\bar{\psi}'\vec{\gamma}\psi'.\nabla\lambda\,.$$

However, the additional last terms just compensates the last $\nabla(x)$ depending term in (2.133).

We see that the QED Hamiltonian can be always chosen in the form (we write ψ and $\bar{\psi}$ instead of ψ' and $\bar{\psi}'$):

$$\begin{aligned} H \;=\; & \int d^3\vec{x}\,[\frac{1}{2}\,\dot{\vec{A}}_\perp^2(x) \;+\; \frac{1}{2}\,(\nabla\times\vec{A}_\perp)^2(x)] \;+\; \int d^3\vec{x}\,\bar{\psi}(x)(-i\vec{\gamma}.\nabla \,+\, m)\psi(x) \\ & -\, e\int d^3\vec{x}\,\vec{J}(x).\vec{A}_\perp(x) \;+\; \frac{e^2}{8\pi}\int d^3\vec{x}d^3\vec{y}\,\frac{\rho(t,\vec{x})\,\rho(t,\vec{y})}{|\vec{x}\,-\,\vec{y}|}\,, \end{aligned} \tag{2.140}$$

The terms in H have the following interpretation:

• The first line in (2.140) corresponds to the Hamiltonian describing free transversal photons and free charged particles (electrons and positrons),

• The second line in (2.140) describes interactions: the first term corresponds to the interaction of photons with charged particles and second one represents the mutual Coulomb interaction of charged particles.

The gauge invariance. The remarkable success with the elimination of the longitudinal magnetic field in QED Hamiltonian, so that were left charged particles and photons, is closely related to the local gauge invariance of QED Lagrangian.

Inn fact, it can easily shown that the QED Lagrangian $\mathcal{L}(A,\psi,\bar{\psi})$ is invariant with respect to the *local gauge transformations*

$$A_\mu(x) \;\mapsto\; A'_\mu(x) \;=\; A_\mu(x) \;+\; \partial_\mu\alpha(x)\,.$$

$$\psi(x) \;\mapsto\; \psi'(x) \;=\; e^{ie\alpha(x)}\,\psi(x)\,, \quad \bar{\psi}(x) \;\mapsto\; \bar{\psi}'(x) \;=\; \bar{\psi}(x)\,e^{-ie\alpha(x)}\,, \tag{2.141}$$

where $\alpha(x)$ generates a real local (x-depending) change of the phase of Dirac field:

• The terms $\frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu}$ and $m\,\bar{\psi}\psi$ are evidently gauge invariant, and

• the changes of the terms $i\bar{\psi}\,\gamma^\mu\,\partial_\mu\psi$ and $-e\,\bar{\psi}\,\gamma^\nu\,\psi\,A_\nu$ mutually cancel under gauge transformations.

Note. A special choice of the phase function $\alpha(x)$ may guarantee that electromagnetic field potential satisfies some additional *gauge condition*. Most often used gauges are the following:

• *The Coulomb gauge* $\nabla.\vec{A} = 0$ in which the electromagnetic potential is transversal. This we were considering above, when we eliminated the longitudinal part of the potential;

• *The Lorentz gauge* $\partial_\mu\, A^\mu = 0$. This is a relativistic invariant gauge condition (which remains same in all inertial systems). Later we shall consider the electromagnetic field propagator in relativistic gauges.

2.3. Perturbation approach to QED

QED in the Coulomb gauge

Below we shall apply perturbation method to the field-theoretic system described by the QED Hamiltonian in Coulomb gague:

$$H = \int d^3\vec{x}\,\bar{\psi}(x)(-i\vec{\gamma}.\nabla + m)\psi(x) + \int d^3\vec{x}\,[\frac{1}{2}\,\dot{\vec{A}}^2_\perp(x) + \frac{1}{2}\,(\nabla\times\vec{A}_\perp)^2(x)]$$
$$- e\int d^3\vec{x}\,\vec{J}(x).\vec{A}_\perp(x) + \frac{e^2}{8\pi}\int d^3\vec{x}d^3\vec{y}\,\frac{\rho(t,\vec{x})\,\rho(t,\vec{y})}{|\vec{x} - \vec{y}|}\,. \qquad (2.142)$$

In the first line we have the Hamiltonian for free Dirac electrons and positrons with mass m and charge $\pm e$ and the Hamiltonian for free transversal photons with vanishing mass. The first term on the second line describes the interaction of charged particles with photons and the second one represents the mutual Coulomb interaction of charged particles.

The fermions in interaction picture. In the interaction picture the electron and positrons are described by free Dirac fields $\psi(x)$ and $\bar{\psi}(x)$ which can be expanded into plane waves

$$\psi(x) \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(b_p^s\, u^s(p)\, e^{-ipx} \;+\; c_p^{s\dagger}\, v^s(p)\, e^{ipx}\right) , \qquad (2.143)$$

$$\bar{\psi}(x) \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2E_p}} \sum_s \left(c_p^s\, \bar{v}^s(p)\, e^{-ipx} \;+\; b_p^{s\dagger}\, \bar{u}^s(p)\, e^{ipx}\right) , \qquad (2.144)$$

where b_p^s and $b_p^{s\dagger}$ are fermionic annihilation and creation operators in Fock describing *electrons* with 4-momentum $p = (E_p, \vec{p})$, $E_p = \sqrt{\vec{p}^2 + m^2}$, and spin $s = \pm 1/2$. Similarly, c_p^s and $c_p^{s\dagger}$ are fermionic annihilation and creation operators describing *positrons*.

External fermion links

Electron in initial state:

$$\psi(x)\,\sqrt{2E_p} b_p^{s\dagger} \;=\; \sqrt{2E_p}\langle 0|\, \psi(x)\, b^{s\dagger_p}\, |0\rangle \;=\; u_s(p)\, e^{ipx} \,. \qquad (2.145)$$

Electron in final state:

$$\sqrt{2E_p} b_p^s\, \bar{\psi}(x) \;=\; \sqrt{2E_p}\langle 0|\, b_p^s\, \bar{\psi}(x)\, |0\rangle \;=\; \bar{u}_s(p)\, e^{-ipx} \,. \qquad (2.146)$$

Positron in initial state:

$$\bar{\psi}(x)\,\sqrt{2E_p} c_p^{s\dagger} \;=\; \sqrt{2E_p}\langle 0|\, \bar{\psi}(x)\, c_p^{s\dagger}\, |0\rangle \;=\; \bar{v}_s(p)\, e^{-ipx} \,. \qquad (2.147)$$

Positron in final state:

$$\sqrt{2E_p} c_p^s\, \psi(x) \;=\; \sqrt{2E_p}\langle 0|\, c_p^s\, \psi(x)\, |0\rangle \;=\; v_s(p)\, e^{ipx} \,. \qquad (2.148)$$

Internal fermion links are given by Feynman propagator:

$$S_F(x-y) \;=\; \langle 0|\, T[\psi(x)\bar{\psi}(y)]\, |0\rangle \;=\; \frac{i}{(2\pi)^4}\int d^4p\, \frac{p.\gamma + m}{p^2 - m^2 + i\varepsilon}\, e^{-ip(x-y)} \,. \qquad (2.149)$$

The fermion contraction $\langle 0|T[\psi(x)\psi(y)]|0\rangle$ and $\langle 0|T[\bar{\psi}(x)\bar{\psi}(y)]\,|0\rangle$ vanish.

Photons in interaction picture. Photons are described as the transversal vector field of zero mass bosons:

$$\vec{A}_\perp(x) \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{\sqrt{2\omega_k}}\sum_\sigma \left(a_k^\sigma\,\vec{e}_\sigma(k)\,e^{-ikx} \;+\; a_k^{\sigma\dagger}\,\vec{e}^{\,*}_\sigma(k)\,e^{ikx}\right)\,, \quad (2.150)$$

where the photon 4-momentum is given as $k \;=\; (\omega_k,\vec{k})$, $\omega_k \;=\; |\vec{k}|$. The complex polarization vectors are orthonomal and transversal $\vec{e}_\sigma(k)\,.\,\vec{k} \;=\; \vec{e}^{\,*}_\sigma(k)\,.\,\vec{k} \;=\; 0$. The photon annihilation and creation operators satisfying bosonic commutation relations:

$$[a_\sigma(k), a^\dagger_{\sigma'}(k')] \;=\; (2\pi)^3\,\delta(\vec{k}-\vec{k},)\,\delta_{\sigma\sigma'}\,. \quad (2.151)$$

The annihilation operators mutually commute, and similarly the creation operators.

External photon links

Photon in initial state:

$$\vec{A}_\perp(x)\,\sqrt{2\omega_k}a_k^{\sigma\dagger} \;=\; \sqrt{\omega_k}\langle 0|\,\vec{A}_\perp(x)\,a_k^{\sigma\dagger}\,|0\rangle \;=\; \vec{e}_\sigma(k)\,e^{-ikx}\,. \quad (2.152)$$

Photon in final state:

$$\sqrt{2\omega_k}a_k^\sigma\,\vec{A}_\perp(x) \;=\; \sqrt{\omega_k}\langle 0|\;a_k^\sigma\,\vec{A}_\perp(x)\,|0\rangle \;=\; \vec{e}^{\,*}_\sigma(k)\,e^{ikx}\,. \quad (2.153)$$

Internal photon links are given by the propagator of transversal photons:

$$\begin{aligned} D_C^{ij}(x-y) \;&=\; \langle 0|\,T[A^i_\perp(x)\,A^j_\perp(y)]\,|0\rangle \\ =\; \frac{1}{(2\pi)^4}&\int d^4k\,\frac{i}{k^2+i\varepsilon}\left(\delta^{ij}-\frac{k^ik^j}{\vec{k}^2}\right)\,e^{-ip(x-y)}\,. \end{aligned} \quad (2.154)$$

Vertices follow from the interaction Hamiltonian:

$$H_{int} \;=\; -\,e\int d^3\vec{x}\,\vec{J}(x).\vec{A}_\perp(x) \;+\; \frac{e^2}{8\pi}\int d^3\vec{x}d^3\vec{y}\,\frac{\rho(t,\vec{x})\,\rho(t,\vec{y})}{|\vec{x}\,-\,\vec{y}|}\,, \quad (2.155)$$

where $\vec{J}(x) = \bar{\psi}(x)\vec{\gamma}\psi(x)$ and $\rho(x) = \bar{\psi}(x)\gamma^0\psi(x)$. The first term describes the interaction of electrons and positrons with photons and the second term the Coulomb interaction among charged particles.

Vertex 1. To the first term in interaction Hamiltonian we assign the expression

$$-\,e\,\gamma^j \int d^4x \,\ldots\,. \tag{2.156}$$

The corresponding diagram is determined by a 1 point vertex at x: to the vertex are attached 2 arrowed fermion links (one arrow oriented *into* vertex and the other *out* of vertex) and a photon link with index j. At the same time we indicated the needed integration over positions of the d^4x.

Vertex 2. To the second term in interaction Hamiltonian we assign the expression

$$\frac{e^2}{8\pi}e \int dt\, d^3\vec{x}\, d^3\vec{y}\, \frac{1}{|\vec{x}-\vec{y}|}\ldots \;=\; \frac{e^2}{8\pi}\int d^4x\, d^4y\, \frac{\delta(x^0-y^0)}{|\vec{x}-\vec{y}|}\ldots\,. \tag{2.157}$$

The corresponding diagram is specified by 2 vertices x and y, to each one are two attached 2 arrowed fermion links (one arrow oriented *into* vertex and the other *out* of vertex). The vertices x and y are connected by corresponding to the integral kernel $\delta(x^0-y^0)/|\vec{x}-\vec{y}|$ responsible for the instantaneous Coulomb interaction. Again we indicated the integrations over d^4x a d^4y.

The full electromagnetic propagator

A way how to simplify considerably the QED diagrammatic rules in Coulomb gauge can be seen by a closer look to the electron elastic scattering amplitude up to the power e^2 in the electric charge - the lowest order of perturbation theory.

We are interesting in the process with 2 electrons both in initial and states:

$$|i\rangle \;=\; |p_1, s_1; p_2, s_2\rangle \;\longrightarrow\; |f\rangle \;=\; |p_1', s_1'; p_2', s_2'\rangle\,. \tag{2.158}$$

We point out that there is no contribution proportional to e. There are to types of contribution proportional to e^2:

• The second order contribution from the first term in interaction Hamiltonian proportional to e e which describes the interaction of charged particles with photons, and

• the first order contribution from the second term in interaction Hamiltonian proportional to e^2 which describes the mutual Coulomb interaction of charged particles.

The Wick theorem gives the following contribution of the order e^2 to the scattering amplitude in question:

$$S_{fi}^{(2)} = \sum{}' \frac{(ie)^2}{2}\langle f| \int d^4x d^4y \ :\bar{\psi}(x)\gamma^i\psi(x):\ D_C^{ij}(x-y)\ :\bar{\psi}(y)\gamma^j\psi(y):\ |i\rangle$$
$$+ \sum{}' (-i)\frac{e^2}{8\pi}\langle f| \int d^4x d^4y \ :\bar{\psi}(x)\gamma^0\psi(x):\ \frac{x^0-y^0}{|\vec{x}-\vec{y}|}\ :\bar{\psi}(y)\gamma^0\psi(y):\ |i\rangle\,. \tag{2.159}$$

In the first term we have explicitly indicated the 2 photon contraction (propagator) $A_\perp^i(x)A_\perp^j(y) = D_C^{ij}(x-y)$. The symbol $\sum'$ denotes the sum over Dirac field contractions between electrons in initial/final states and those in currents (represented by external lines). We stress that those summations are *identical* in both interaction terms:

• The contributions from the first term have the form,

$$e^2\ :\bar{\psi}(x)\gamma_i\psi(x):\ D_C^{ij}(x-y)\ :\bar{\psi}(y)\gamma_j\psi(y):, \tag{2.160}$$

• whereas, in the second term we a very similar expression

$$e^2\ :\bar{\psi}(x)\gamma_0\psi(x):\ D_C^{00}(x-y) :\bar{\psi}(y)\gamma_0\psi(y):, \tag{2.161}$$

in which γ_i and γ_j are replaced by two γ_0's and $D_C^{ij}(x-y)$ is replaced by

$$D_C^{00}(x-y) \equiv \frac{-i}{4\pi}\frac{\delta(x^0-y^0)}{|\vec{x}-\vec{y}|}\,., \tag{2.162}$$

From the structure of QED interaction Hamiltonian (2.155) is evident, that regardless the process in question and the order of perturbation expansion, it holds: the contribution with s $D_F^{ij}(x-y)$ is supplemented by the same expression with $D_F^{00}(x-y)$.

The photon propagator in Coulomb gauge. The analysis presented above suggest the following modification of Feynman rules:

Internal photon links. We join the transversal photon propagator $D^{ij}_C(x-y)$ with $D^{00}_C(x-y)$ into one common Feynman photon propagator which in x-representation in Coulomb gauge possesses the $D^{\mu\nu}_C(x-y)$, $\mu,\nu = 0,1,2,3$ given as follows:

$$D^{00}_C(x-y) = \frac{-i}{4\pi}\frac{\delta(x^0-y^0)}{|\vec{x}-\vec{y}|} = \frac{1}{(2\pi)^4}\int d^4k\, \frac{-i}{\vec{k}^2}\, e^{-ik(x-y)}\,,$$

$$D^{0j}_C(x-y) = D^{j0}_C(x-y) = 0\,,$$

$$D^{ij}_C(x-y) = \frac{1}{(2\pi)^4}\int d^4k \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right)\frac{i}{k^2+i\varepsilon}\, e^{-ik(x-y)}\,. \qquad (2.163)$$

In p-representation the photon propagator in Coulomb gauge is given as:

$$\tilde{D}^{00}_C(k) = \frac{-i}{\vec{k}^2}\,, \qquad \tilde{D}^{0j}_C(k) = \tilde{D}^{j0}_C(k) = 0\,,$$

$$\tilde{D}^{ij}_C(k) = \frac{i}{k^2+i\varepsilon}\left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right)\,. \qquad (2.164)$$

Note: In the process of the derivation of integral representation of $D^{00}_C(x-y)$ we have used the well-known formula

$$\delta(x^0-y^0) = \frac{1}{2'pi}\int dk^0\, e^{-ik^0(x^0-y^0)}\,, \quad \frac{1}{|\vec{x}-\vec{y}|} = \frac{1}{2\pi^2}\int d^3\vec{k}\,\frac{1}{\vec{k}^2}\, e^{i\vec{k}.(\vec{x}-\vec{y})}$$

Vertex v QED. The propagator (2.163) (or (2.164) in $p$representation) corresponds to *one* vertex

$$-\,e\,\gamma^\mu \int d^4x\ \ldots\,. \qquad (2.165)$$

We assign to this vertex the diagram specified by one position x: to vertex are attached 2 arrowed fermion links (one arrow oriented *into* vertex and the other *out* of vertex) and a photon link with index $\mu = 0,1,2,3$. At the same time we indicated the needed integration over positions of the d^4x.

The vertex (2.165) would follow from the interaction Lagrangian density

$$\mathcal{L}_{int} \;=\; -e \;:\bar{\psi}(x)\gamma^{\mu}\psi(x)\,A_{\mu}(x):\;. \tag{2.166}$$

Effectively, the Coulomb interaction among charged particles was replaced by a "new" interaction mediated by longitudinal electromagnetic field. This field just generates the Coulomb interaction among charged particles and there no "new" particles - longitudinal photons in initial and final states. The remaining Feynman rules stay unchanged:
• The rule for external and internal fermion links, and
• the rule for external transversal photon links.

Gauge transformation of photon propagator. We prove that the perturbation contributions to scattering amplitude are unchanged under *gauge transformation* of photon propagator:

$$D_C^{\mu\nu}(x-y) \;\mapsto\; D^{\mu\nu}(x-y) \;\equiv\; D_C^{\mu\nu}(x-y) \,+\, \partial^{\mu}\chi^{\nu}(x-y) \,+\, \partial^{\mu}\chi^{\nu}(x-y)\,, \tag{2.167}$$

where $\chi^{\mu}(z)$ is an arbitrary function of the variable $z = x - y$.

This is a consequence of the fact that the propagator $D_C^{\mu\nu}(x-y)$ *always* enters the perturbation contributions in the following way:

$$\int d^4x \ldots d^4y \;:\bar{\psi}(x)\gamma_{\mu}\psi(x):\;\ldots\; D_C^{\mu\nu}(x-y)\;\ldots\;:\bar{\psi}(y)\gamma^{j}\psi(y):\;. \tag{2.168}$$

Performing the gauge transformation (2.167) the integral (2.168) gains two similar terms:
• The first one vanishes

$$\int d^4x \ldots d^4y \;:\bar{\psi}(x)\gamma_{\mu}\psi(x):\;\partial^{\mu}\chi^{\nu}(x-y)\;\ldots\;:\bar{\psi}(y)\gamma_{\nu}\psi(y):$$

$$=\; -\int d^4x \ldots d^4y \;:\partial^{\mu}(\bar{\psi}(x)\gamma_{\mu}\psi(x)):\;\chi^{\nu}(x-y)\;\ldots\;:\bar{\psi}(y)\gamma_{\nu}\psi(y):\;=\;0\,.$$

Here we have performed per-partes integration over d^4x (under assumption that the boundary $x \to \infty$ does not contribute) and then we used the continuity equation for the electromagnetic current $\partial^{\mu}(\bar{\psi}(x)\gamma_{\mu}\psi(x)) = 0$.
• The first one vanishes due to similar reasons

$$\int d^4x \ldots d^4y \;:\bar{\psi}(x)\gamma_{\mu}\psi(x):\;\partial^{\nu}\chi^{\mu}(x-y)\;\ldots\;:\bar{\psi}(y)\gamma_{\nu}\psi(y):$$

$$= \; - \int d^4x \ldots d^4y \;:\bar{\psi}(x)\gamma_\mu\psi(x):\; \chi^\mu(x-y)\;\ldots\;:\partial^\nu(\bar{\psi}(y)\gamma_\nu\psi(y)):\; = \; 0\,.$$

In p-representation the gauge transformation (2.167) of the photon propagator is given as

$$\tilde{D}_C^{\mu\nu}(k) \;\rightarrow\; \tilde{D}^{\mu\nu}(k) \;=\; \tilde{D}_C^{\mu\nu}(k) \,+\, k^\mu\,\tilde{\chi}^\nu(k) \,+\, k^\nu\,\tilde{\chi}^\mu(k)\,, \tag{2.169}$$

where $\tilde{\chi}^\mu(k)$ is the Fourier transform of the function $\chi^\mu(z)$.

The photon propagator in Feynman gauge. Let us choose the function $\tilde{\chi}^\mu(k)$ as follows:

$$\tilde{\chi}^0(k) \;=\; \frac{ik^0}{k^2\,\vec{k}^2}\,, \quad \tilde{\chi}^j(k) \;=\; \frac{-k^j}{k^2\,\vec{k}^2}\,. \tag{2.170}$$

Performing the corresponding p-representation gauge transformation (2.169), the propagator $\tilde{D}_C^{\mu\nu}(k)$ is mapped to a simple fully relativistic form - the photon propagator in *Feynman gauge*:

$$\tilde{D}_F^{\mu\nu}(k) \;=\; \frac{-i\,\eta_{\mu\nu}}{k^2\,+\,i\varepsilon}\,. \tag{2.171}$$

In x-representation the Feynman propagator represented by the function

$$D_F^{\mu\nu}(x-y) \;=\; \frac{1}{(2\pi)^4}\int\,,d^4k\,\frac{-i\,\eta_{\mu\nu}}{k^2\,+\,i\varepsilon}\,e^{-ik(x-y)}\,. \tag{2.172}$$

In Feynman gauge the vertex is represented by the diagram with 1 vertex to which are attached 2 arrowed fermion links (one arrow oriented *into* vertex and the other *out* of vertex) and a photon link with index μ. To vertex we assign the expression In p-representation

$$-i\,e\,\gamma^\mu\,. \tag{2.173}$$

• In p-representation we assign particle 4-momentum to any link attached to the vertex - these momenta satisfy the 4-momentum conservation law in the vertex,

• In x-representation we assign to vertex its position x - the integration over d^4x is assumed.

The Feynman gauge represented a great progress: the formalism is relativistic invariant and the rules for diagram construction are much simpler than those in Coulomb gauge.

Note: One can use other relativistic gauges for photon propagator. Quite popular is the relativistic gauge depending on one parameter ξ:

$$\tilde{D}_{\xi}^{\mu\nu}(k) \; = \; \frac{-i}{k^2 \, + \, i\varepsilon} \left(\eta_{\mu\nu} \, - \, \xi \, \frac{k^{\mu} k^{\nu}}{k^2} \right) \, . \tag{2.174}$$

The case $\xi = 0$ corresponds to Feyman gauge, whereas $\xi = 1$ corresponds to Landau gauge.

The rules for the calculation of Feynman diagrams for self-interacting scalar field and QED in Feynman gauge are summarized in attached Table.