YET ANOTHER INTRODUCTION TO

QUANTUM FIELD THEORY

WITH THE EMPHASIS ON

THE REAL UNDERSTANDING OF VARIOUS NOTIONS

INCLUDING, BUT NOT RESTRICTED TO:

THE LOGIC OF THE SUBJECT

CANONICAL QUANTIZATION, PATH INTEGRALS

FEYNMAN DIAGRAMS

RENORMALIZATION AND REGULARIZATION

FROM TOY MODELS UP TO THE STANDARD MODEL

FOR THE USE OF STUDENTS

AND PARTICLE PHYSICISTS IN GENERAL

WITH AN APPENDIX ON SOLID STATE PHYSICS.

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ENJOY.

Contents

Preface

When I started giving lectures on Quantum field theory around 2000, I had no intention to write a book on the subject. There are plenty of such books available on the market and it seemed to make a little sense to add another one. But, unfortunately, none of them was exactly to my taste. My favorite *The Quantum Theory of Fields* by Steven Weinberg did not suit that well as an introductory course. I decided to use *An Introduction to Quantum Field Theory* by Peskin and Schroeder, which was, and perhaps still is, one of the standard modern textbooks on the subject. The book, however, leaves much to be desired, so I started to write some notes to provide a set of hopefully useful comments and remarks to it. The original plan was

- to reorganize the material in a bit different way
- to offer sometimes a slightly different point of view
- to add some material

Eventually, the text became more and more self-contained, and the resemblance to the Peskin-Schroeder became weaker and weaker. At the present point, the text has very little to do with the Peskin-Schroeder, except perhaps the largely common notation.

The aim of this course is to explain not only what we are doing, but why we are doing it. I tried my best not to provide complicated answers to questions that were not asked. This applies not only to particular aspects of the subject, but also to the structure of the whole course.

In the quite extended first part almost no interesting particle physics is discussed at all. We only deal with scalar fields and spinless particles, the emphasis is on the logic of the theory (with all the necessary technicalities, of course). In this part students should learn and understand why and how do we quantize classical fields, why and how the machinery of Feynman diagrams works, why and how do we renormalize parameters of lagrangians, why and how do we utilize path integral formulation of QFT.

The physics comes in only in the second part, devoted to Quantum Electrodynamics. Here the technical complications brought up by higher spins, as well as important physical results are discussed thoroughly. All this is done step by step. We start with spinless particles in classical electromagnetic field, then the QED of spinless particles is developed, and only afterwards the full (spinor) QED appears.

The third part concerns the Standard Model. Large portion of this part, however, does not deal with the SM itself, but rather with the particle physics before the SM. It is my firm belief, that students exposed directly to the SM lagrangian, with insufficient knowledge of the prior theoretical (and experimental) development, can miss the essence of the whole business. But it is not only the historic perspective what makes the pre-SM particle physics very useful for the SM course. Virtually all the ingredients of the SM originated in pre-SM physics and so they can be introduced in a quite natural way. Only once these ingredients are grasped to a reasonable level, the SM is discussed.

Part I

Basics of QFT (Spinless Particles)

Chapter 1

Introductions

Let us state at the very beginning that Quantum field theory is

- a theory of **particles** (in the way Quantum mechanics is a theory of atoms and molecules)
- mathematically ill-defined
- the most precise theory mankind ever had
- conceptually and technically quite demanding

Mainly because of the last feature, it seems reasonable to spend enough time with introductions. The reason for plural is that we shall try to introduce the subject in couple of different ways.

Our first introduction is in fact a summary. We shall try to show how QFT is used in practical calculations, without any attempt to understand why it is used in this way. The reason for this strange maneuver is that, surprisingly enough, it is much easier to grasp the bulk of QFT on this operational level than to really understand it. We believe that even a superficial knowledge of how QFT is usually used can be quite helpful in a subsequent, more serious, study of the subject.

The second introduction is a brief exposition of the nonrelativistic many-particle quantum mechanics. This enables a natural introduction of many basic ingredients of QFT (the Fock space, creation and annihilation operators, calculation of vacuum expectation values, etc.) and simultaneously to avoid the difficult question of merging relativity and quantum theory.

It is the third introduction, which sketches that difficult question (merging relativity and quantum theory) and this is done in the spirit of the Weinberg's book. Without going into technical details we try to describe how the notion of a relativistic quantum field enters the game in a natural way. The main goal of this third introduction is to clearly formulate the question, to which the canonical quantization provides an answer.

Only then, after these three introductions, we shall try to develop QFT systematically. Initially, the development will concern only the scalar fields (spinless particles). More realistic theories for particles with spin 1/2 and 1 are postponed to later chapters.

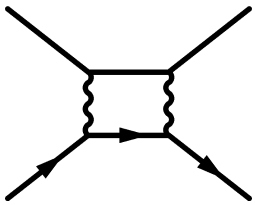
1.1 Conclusions

The machinery of QFT works like this:

- typical formulation of QFT — specification of a Lagrangian $\mathcal{L}$
- typical output of QFT — cross-sections $d\sigma/d\Omega$
- typical way of obtaining the output — Feynman diagrams

The machinery of Feynman diagrams works like this:

- For a given process (particle scattering, particle decay) there is a well defined set of pictures (graphs, diagrams). The set is infinite, but there is a well defined criterion, allowing for identification of a relatively small number of the most important diagrams. Every diagram consists of several types of lines and several types of vertices. The lines either connect vertices (internal lines, propagators) or go out of the diagrams (external legs). As an example we can take



- Every diagram has a number associated with it. The sum of these numbers is the so-called scattering amplitude. Once the amplitude is known, it is straightforward to obtain the cross-section — one just plugs the amplitude into a simple formula.
- The number associated with a diagram is the product of factors corresponding to the internal lines, external lines and the vertices of the diagram. Which factor corresponds to which element of the diagram is the content of the so-called Feynman rules. These rules are determined by the Lagrangian.
- The whole scheme is

Lagrangian
↓
Feynman rules
↓
Feynman diagrams
↓
scattering amplitude
↓
cross-section

- Derivation of the above scheme is a long and painful enterprise. Surprisingly enough, it is much easier to formulate the content of particular steps than to really derive them. And this formulation (without derivation[1]) is the theme of our introductory summary.

[1] It is perhaps worth mentioning that the direct formulation (without derivation) of the above scheme can be considered a fully sufficient formulation of the real content of QFT. This point of view is advocated in the famous Diagrammar by Nobel Prize winners 't Hooft and Veltman, where *"corresponding to any Lagrangian the rules are simply defined"*

1.1.1 Feynman rules

The role of the Lagrangian in QFT may be a sophisticated issue, but for the purposes of this summary the Lagrangian is just a set of letters containing the information about the Feynman rules. To decode this information one has to know, first of all, which letters represent fields (to know what the word *field* means is not necessary). For example, in the toy-model Lagrangian

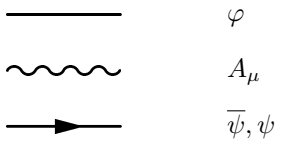
$$\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{g}{3!}\varphi^3$$

the field is represented by the letter φ. Other symbols are whatever but not fields (as a matter of fact, they correspond to space-time derivatives, mass and coupling constant, but this is not important here). Another example is the Lagrangian of quantum electrodynamics (QED)

$$\mathcal{L}\left[\overline{\psi},\psi,A_\mu\right] = \overline{\psi}\left(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the fields are $\overline{\psi}$, ψ and A_μ (the symbol γ^μ stands for the so-called Dirac matrices, and ξ is called a gauge parameter, but this information is not relevant here).

Now to the rules. Different fields are represented by different types of lines. The usual choice is a simple line for φ (called the scalar field), the wiggly line for A_μ (called in general the massless vector field, in QED the photon field) and a simple line with an arrow for $\overline{\psi}$ and ψ (called in general the spinor field, in QED usually the electron-positron field).



The arrows are commonly used for complex fields, like $\overline{\psi}$ and ψ (or φ^* and φ, if φ is complex)[2]. The arrow orientation is very important for external legs, where different orientations correspond to particles and antiparticles respectively (as we will see shortly).

Every line is labelled by a momentum (and perhaps some other quantum numbers). The arrows discussed above and their orientation do not represent the momentum associated with the line!

The Feynman rules associate a specific factor with every internal line (propagator), line junction (vertex) and external line. Propagators are defined by the part of the Lagrangian quadratic in fields. Vertices are given by the rest of the Lagrangian. External line factor depends on the whole Lagrangian and usually (but not necessarily) it takes a form of the product of two terms. One of them is simple and is fully determined by the field itself, i.e. it does not depend on the details of the Lagrangian, while the other one is quite complicated.

[2]Actually, in practice arrows are not used for scalar field, even if it is complex. The reason is that no factors depend on the arrows in this case, so people just like to omit them (although in principle the arrows should be present).

vertices

For a theory of one field φ, the factor corresponding to the n-leg vertex is[3]

$$n\text{-leg vertex} = i\left.\frac{\partial^n \mathcal{L}}{\partial \varphi^n}\right|_{\varphi=0}$$

For a theory with more fields, like QED, the definition is analogous, e.g. the vertex with l, m and n legs corresponding to $\overline{\psi}, \psi$ and A_μ-fields respectively, is

$$(l,m,n)\text{-legs vertex} = i\left.\frac{\partial^{l+m+n}\mathcal{L}}{\partial A_\mu^l \partial \psi^m \partial \overline{\psi}^n}\right|_{\text{fields}=0}$$

Each derivative with respect to φ produces a corresponding leg entering the vertex. For terms containing space-time derivative of a field, e.g. $\partial_\mu\varphi$, the derivative with respect to φ is defined in a bit bizarre way as[4]

$$\frac{\partial}{\partial\varphi}\partial_\mu\varphi \times \text{something} = -ip_\mu \times \text{something} + \partial_\mu\varphi \times \frac{\partial}{\partial\varphi}\text{something}$$

p_μ is the momentum (towards the vertex) of the leg produced by this derivative.

Clearly, examples are called for. In our toy-model given above (the so-called φ^3-theory) the non-quadratic part of the Lagrangian contains the third power of the field, so there will be only the 3-leg vertex

$$= \quad i\frac{\partial^3}{\partial\varphi^3}\left(-\frac{g}{3!}\varphi^3\right) = -ig$$

In our second example, i.e. in QED, the non-quadratic part of the Lagrangian is $-\overline{\psi}q\gamma^\mu A_\mu\psi$, leading to the single vertex

$$= \quad i\frac{\partial^3\left(-q\overline{\psi}\gamma^\mu A_\mu\psi\right)}{\partial\overline{\psi}\partial\psi\partial A_\mu} = -iq\gamma^\mu$$

and for purely didactic purposes, let us calculate the vertex for the theory with the non-quadratic Lagrangian $-g\varphi^2\partial_\mu\varphi\partial^\mu\varphi$

$$\begin{aligned}
&= \quad i\frac{\partial^4}{\partial\varphi^4}\left(-g\varphi^2\partial_\mu\varphi\partial^\mu\varphi\right)\\
&= -ig\frac{\partial^3}{\partial\varphi^3}\left(2\varphi\partial_\mu\varphi\partial^\mu\varphi - 2i\varphi^2 p_1^\mu\partial_\mu\varphi\right)\\
&= -ig\frac{\partial^2}{\partial\varphi^2}\left(2\partial_\mu\varphi\partial^\mu\varphi - 4i\varphi p_2^\mu\partial_\mu\varphi - 4i\varphi p_1^\mu\partial_\mu\varphi - 2\varphi^2 p_1^\mu p_{2,\mu}\right)\\
&= -i4g\frac{\partial}{\partial\varphi}\left(-ip_3^\mu\partial_\mu\varphi - ip_2^\mu\partial_\mu\varphi - \varphi p_2 p_3 - ip_1^\mu\partial_\mu\varphi - \varphi p_1 p_3 - \varphi p_1 p_2\right)\\
&= 4ig\left(p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4\right)
\end{aligned}$$

[3] The RHS of this definition could (should) contain a factor $(2\pi)^4\,\delta^4\left(p_1 + p_2 + \ldots + p_n\right)$ where p_i is the momentum corresponding to the i-th leg (all momenta are understood to be pointing towards the vertex). However, we prefer to include this factor elsewhere.

[4] $\frac{\partial}{\partial\varphi}\partial_\mu\varphi$ is by definition equal to $-ip_\mu$, and the Leibniz rule applies to $\frac{\partial}{\partial\varphi}$, as it should apply to anything worthy of the name derivative.

propagators

Propagators are defined by the quadratic part of the Lagrangian. They are negative inverses of the 2-leg vertices with an ε adornment (and with momenta p and $p' = -p$ pointing towards the vertex)

$$\text{propagator} = i\left(\left.\frac{\partial^2\mathcal{L}}{\partial\varphi^2}\right|_{\varphi=0,p'=-p} + i\varepsilon\right)^{-1}$$

The symbol ε stands for any positive infinitesimal quantity, therefore we will always replace $\varepsilon \times$ finite quantity by the ε itself. For complex fields one uses $\partial^2\mathcal{L}/\partial\varphi^*\partial\varphi$, definitions for other fields are similar.

The examples below are more than examples, they are universal tools to be used over and over. The point is that the quadratic parts of Lagrangians are the same in almost all theories, so once the propagators are calculated, they can be used in virtually all QFT calculations.

The quadratic part of the scalar field Lagrangian is $\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$, leading to $\partial^2\mathcal{L}/\partial\varphi^2|_{\varphi=0,p'=-p} = -p.p' - m^2|_{p'=-p} = p^2 - m^2$, i.e.

$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

The quadratic part of the spinor field Lagrangian is $\overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$, leading to $\partial^2\mathcal{L}/\partial\overline{\psi}\partial\psi|_{\text{fields}=0,p'=-p} = \gamma^\mu p_\mu - m$, i.e.

$$= \frac{i}{\gamma^\mu p_\mu - m + i\varepsilon} = \frac{i\left(\gamma^\mu p_\mu + m\right)}{p^2 - m^2 + i\varepsilon}$$

where we have utilized the identity $\left(\gamma^\mu p_\mu - m\right)\left(\gamma^\mu p_\mu + m\right) = p^2 - m^2$, which at this stage is just a God-given identity, allowing to write the propagator in the standard way with $p^2 - m^2 + i\varepsilon$ in the denominator.

Finally, for the massless vector field the quadratic Lagrangian is $-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2\xi}\left(\partial_\alpha A^\alpha\right)^2$ leading to[5,6,7] $\partial^2\mathcal{L}/\partial A_\mu\partial A_\nu|_{\text{fields}=0,p'=-p} = (1 - \frac{1}{\xi})p^\mu p^\nu - p^2 g^{\mu\nu}$

$$= \frac{i}{\left(1 - \frac{1}{\xi}\right)p^\mu p^\nu - p^2\eta^{\mu\nu} + i\varepsilon} = \frac{-i\left(\eta_{\mu\nu} - (1-\xi)\,p_\mu p_\nu/p^2\right)}{p^2 + i\varepsilon}$$

Surprisingly enough, this is almost everything one would ever need as to the propagators. In the Standard Model, the spinor propagator describes quarks and leptons, the massless vector propagator describes photon and gluons, the scalar propagator describes the Higgs boson. The only missing propagator is the massive vector one, describing the $W^\pm$ and Z^0 bosons. This can be, however, worked out easily from the Lagrangian $-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}m^2A_\alpha A^\alpha$ (the result is $-i\left(\eta^{\mu\nu} - p^\mu p^\nu/m^2\right)\left(p^2 - m^2 + i\varepsilon\right)^{-1}$, the derivation is left as an exercise).

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[5] For $\mathcal{L} = \frac{1}{2}\left[\left(\partial_\alpha A_\beta\right)\left(\partial^\beta A^\alpha\right) - \left(\partial_\alpha A_\beta\right)\left(\partial^\alpha A^\beta\right) - \frac{1}{\xi}\left(\partial_\alpha A^\alpha\right)\left(\partial_\beta A^\beta\right)\right]$ one has $\frac{\partial\mathcal{L}}{\partial A_\mu} = -i\left(p_\alpha\eta^\mu_\beta\partial^\beta A^\alpha - p_\alpha\eta^\mu_\beta\partial^\alpha A^\beta - \frac{1}{\xi}p_\alpha\eta^{\alpha\mu}\partial_\beta A^\beta\right) = -i\left(p_\alpha\partial^\mu A^\alpha - p_\alpha\partial^\alpha A^\mu - \frac{1}{\xi}p^\mu\partial_\beta A^\beta\right)$ and $\frac{\partial^2\mathcal{L}}{\partial A_\mu\partial A_\nu} = -p_\alpha p'^\mu\eta^{\alpha\nu} + p_\alpha p'^\alpha\eta^{\mu\nu} + \frac{1}{\xi}p^\mu p'_\beta\eta^{\beta\nu} = p.p'\eta^{\mu\nu} - p'^\mu p^\nu + \frac{1}{\xi}p^\mu p'^\nu$

[6] To find the matrix inverse to $M^{\lambda\mu}(1-\xi^{-1})p^\lambda p^\mu - p^2\eta^{\lambda\mu}$ one may either make an educated guess $M^{-1}_{\mu\nu} = A\eta_{\mu\nu} + Bp_\mu p_\nu$ (there is nothing else at our disposal) and solve for A and B, or one may simply check that $[(1-\xi^{-1})p^\lambda p^\mu - p^2\eta^{\lambda\mu}]\left(-\eta_{\mu\nu} + (1-\xi)\,p_\mu p_\nu/p^2\right) = \eta^\lambda_\nu p^2$.

[7] Let us remark that without the term $\frac{1}{2\xi}\left(\partial_\alpha A^\alpha\right)^2$ the propagator would not exist, since the 2-leg vertex would have no inverse. Two specific choices of the parameter ξ are known as the Feynman gauge ($\xi = 1$) and the Landau gauge ($\xi = 0$).

external legs

The factor corresponding to an external leg is, as a rule, the product of two factors. Let us start with the simpler one. For the scalar field φ (representing a particle with zero spin) this factor is the simplest possible, it equals to 1. For other fields (representing particles with higher spins) there is a nontrivial first factor for each external leg. This factor is different for particles and antiparticles. It also distinguishes between ingoing and outgoing particles (i.e. between the initial and the final state). The factor depends on the particle momentum and spin, but we are not going to discuss this dependence in any detail here.

As to the massless vector field A_μ (e.g. for the photon, where antiparticle = particle) this factor is

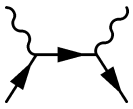
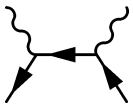
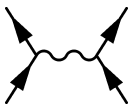
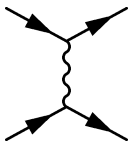
ingoing particle	ε_μ
outgoing particle	ε_μ^*

For the spinor field (e.g. for the electron and positron, which are distinguished in diagrams by the orientation of the arrow) the factor is

ingoing particle	arrow towards the diagram	u
ingoing antiparticle	arrow out of the diagram	$\overline{v}$
outgoing particle	arrow out of the diagram	$\overline{u}$
outgoing antiparticle	arrow towards the diagram	v

These rules are universal, independent of the specific form of the Lagrangian.

Examples for electrons and photons may illuminate the general rules. We will draw diagrams from the left to the right, i.e. ingoing particles (initial state) are on the left and outgoing particles (final state) on the right[8].

process	typical diagram	first external legs factors
$e^-\gamma \to e^-\gamma$		$u, \varepsilon_\mu, \overline{u}, \varepsilon_\nu^*$
$e^+\gamma \to e^+\gamma$		$\overline{v}, \varepsilon_\mu, v, \varepsilon_\nu^*$
$e^+e^- \to e^+e^-$		$\overline{v}, u, v, \overline{u}$
$e^-e^- \to e^-e^-$		$u, u, \overline{u}, \overline{u}$

[8]Note that some authors, including Peskin-Schroeder, draw the Feynman diagrams other way round, namely from the bottom to the top.

Now to the second factor corresponding to an external leg. It has a pretty simple appearance, namely it equals to $\sqrt{Z}$, where Z is a constant (the so-called wave-function renormalization constant) dependent on the field corresponding to the given leg. The definition and calculation of Z are, however, anything but simple.

Fortunately, the dominant part of vast majority of cross-sections and decay rates calculated by means of Feynman diagrams is given by the so-called tree diagrams (diagrams containing no closed loops), and at the tree level the Z constant is always equal to 1. So while staying at the tree level, one can forget about Z completely. And since our first aim is to master the tree level calculations, we can ignore the whole Z-affair until the discussion of loops and renormalization. The following sketch of the Z definition is presented only for the sake of completeness (and can be skipped safely at this moment).

Unlike all other Feynman rules, the Z constant is defined not directly via the Lagrangian, but rather via an infinite sum of Feynman diagrams[9]. The said sum, called the dressed propagator, contains all diagrams with two external legs corresponding to the field under consideration. These two external legs are treated in a specific way — the corresponding factor is not the external leg factor but rather the propagator. The dressed propagator is a function of the external leg momentum (both legs have the same momentum due to the vertex momentum δ-functions) and, as a rule, has a pole in the p^2-variable. The residuum at this pole is the wanted Z.

This definition, as it stands, applies only to the scalar fields. For higher spins the dressed propagator is a matrix and the Z constant is defined via the eigenvalues of this matrix. So one can have, in principle, several different Z constants corresponding to one field. For the electron-positron field, however, there turns out to be only one such constant and the same is true for the photon field.

In addition to this, there is yet another very important ingredient in the external leg treatment. The external leg factor stands not only for the simple (bare) external leg, but rather for the dressed external leg (with all loop corrections). In other words, when calculating a scattering amplitude, one should not include diagrams with loop corrections on external legs. These diagrams are, in a sense, taken into account via the $\sqrt{Z}$ factors[10].

Too complicated? Never mind. Simply forget everything about Z, it will be sufficient to recall it only much later, when dealing with renormalization.

Remark: *As we have indicated, in some circumstances the external leg factor may be even more complicated than the product of two terms (one of them being $\sqrt{Z}$). This happens when there are non-vanishing sums of all diagrams with two external legs corresponding to* different *fields. This is only rarely the case and always indicates that our choice of fields was not the most appropriate one. The remedy for this trouble is quite ordinary: after a suitable re-definition (just a simple linear combination) of the fields, the trouble simply drops out.*

[9] For the defininition of Feynman diagrams see the next subsection.

[10] After being forced to calculate the loop corrections to a simple line in order to obtain Z, one does not need to calculate them again when calculating the scattering amplitude. There is at least some justice in this world.

1.1.2 Feynman diagrams

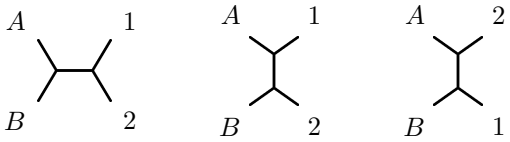
diagrams for a given process contributing at a given order

A process defines external legs, both ingoing and outgoing. A Feynman diagram corresponding to this process is any diagram (graph) with this set of external legs interconnected by the internal lines (propagators) of the theory, via the vertices of the theory, with exception of:

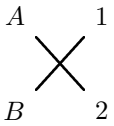
- diagrams with no vertices at all
- diagrams containing so-called "vacuum bubbles", i.e. subdiagrams not connected to any external leg
- diagrams containing so called "corrections on external legs", i.e. subdiagrams with two outgoing lines, one of which is an external leg

There is usually an infinite number of such diagrams. Still, only a finite number contribute at a given order. The order may be defined in at least three different ways, namely as **a**) the number of vertices, **b**) the power of the coupling constant or **c**) the number of (independent) loops. If there is only one type of vertex in the theory, these three definitions are equivalent[11]. If one has more types of vertices, but all characterized by the same coupling constant[12], then the first definition is not used and the other two are not equivalent.

As an example, let us consider a scattering $AB \to 12$, described by either φ^3- or φ^4-theory. At the leading order (the lowest nonzero order, tree level) one has for the φ^3-theory



while for the φ^4-theory

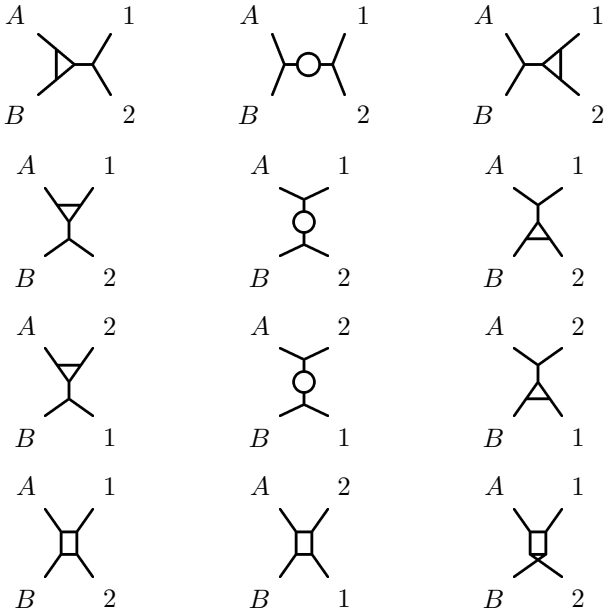


Note that the second and the third diagrams for the φ^3-theory are not equivalent, they contain different vertices (intersections of different lines).

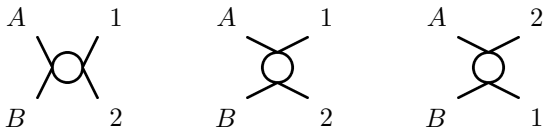
[11] Proof: The equivalence of the first two definitions is evident (every vertex contributes by the same coupling constant). As to the equivalence of the third definition, let us denote the number of vertices, internal lines, external lines and independent loops by V, I, E and L respectively. The famous Euler's Theorem states $V = I - L + 1$. This is to be combined with $nV = 2I + E$ where n is the number of legs of the vertex of the theory. The latter equation is nothing but a simple observation that we can count the number of lines by counting vertices and multiplying their number by n, but we are double-counting internal lines in this way. When combined, the equations give $(n-2)\,V = 2L + E - 2$, i.e. for a given E the dependence of V on L is linear.

[12] A good example is the so-called scalar electrodynamics (spinless particles and photons) defined by the Lagrangian $\mathcal{L}[\varphi] = (D_\mu \varphi)^* D_\mu \varphi - m^2 \varphi^* \varphi$, where $D_\mu = \partial_\mu + iqA_\mu$.

At the next to leading order (1-loop level) one has for the φ^3-theory

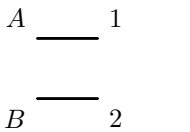


and for the φ^4-theory

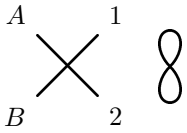


Note that in the last diagram for the φ^3-theory the crossing of external legs B and 2 does not represent a vertex (we just did not manage to draw the diagram in plane without crossing).

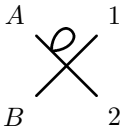
As examples of diagrams not included among Feynman diagrams corresponding to the process under consideration let us mention



no vertices at all



vacuum bubble subdiagram



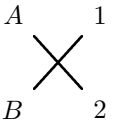
correction on external leg

the factor corresponding to a given diagram

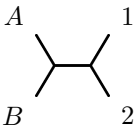
The factor corresponding to a diagram is the product[13] of factors corresponding to all external lines, internal lines and vertices of the diagram, multiplied by

- an extra factor $(2\pi)^4 \delta^4 (p_1 + p_2 + \ldots + p_n)$ for each vertex (p_i is the momentum corresponding to the i-th leg, pointing towards the vertex).
- an extra factor $\int \frac{d^4k}{(2\pi)^4}$ for each propagator (with the four-momentum k)
- an extra so-called combinatorial factor, to be discussed later
- some extra factors of (-1) related to fermionic lines[14]

Examples[15]:

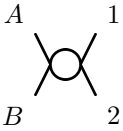


$$= -ig (2\pi)^4 \delta^4 (p_A + p_B - p_1 - p_2)$$



$$= -g^2 (2\pi)^8 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \delta^4 (p_A + p_B - k) \delta^4 (k - p_1 - p_2)$$

$$= -i \frac{g^2}{(p_A+p_B)^2-m^2} (2\pi)^4 \delta^4 (p_A + p_B - p_1 - p_2)$$

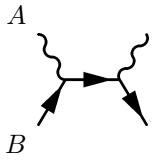


$$= -g^2 (2\pi)^8 \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{k'^2-m^2} \delta^4 (p_A + p_B - k - k') \times$$

$$\uparrow$$

combinatorial factor $\times \delta^4 (k + k' - p_1 - p_2)$

$$= \frac{1}{2} g^2 \int d^4k \frac{1}{k^2-m^2} \frac{1}{(p_A+p_B-k)^2-m^2} \delta^4 (p_A + p_B - p_1 - p_2)$$



$$= -q^2 (2\pi)^8 \int \frac{d^4k}{(2\pi)^4} \overline{u}_B \gamma^\mu \delta^4 (p_A + p_B - k) \frac{i(\gamma^\lambda k_\lambda + m)}{k^2-m^2} \times$$

$$\times \gamma^\nu \delta^4 (k - p_1 - p_2) u_2 \varepsilon^*_{1,\mu} \varepsilon_{A,\nu}$$

$$= -iq^2 \frac{\overline{u}_B \gamma^\mu (\gamma^\lambda (p_A+p_B)_\lambda + m) \gamma^\nu u_2}{(p_A+p_B)^2-m^2} \varepsilon^*_{1,\mu} \varepsilon_{A,\nu} (2\pi)^4 \delta^4 (p_A + p_B - p_1 - p_2)$$

[13] If individual factors are simple numbers, one does not care about their ordering. In some cases, however, these factors are matrices, and then the proper ordering is necessary. The basic rule here is that for every line with an arrow, the factors are to be ordered "against the arrow", i.e. starting from the end of the arrow and going in the reverse direction.

[14] The factor (-1) for every closed fermionic loop, and the relative minus sign for the diagrams, which can be obtained from each other by an interchange of two fermionic lines. Diagrams related to each other by the omission or addition of boson lines have the same sign.

[15] All momenta are understood to flow from the left to the right. One can, of course, choose another orientation of momenta and change the signs in δ-functions correspondingly.

There is another, a bit more economic way, of producing the factor corresponding to a given diagram. When using this method, diagrams are drawn in such a way that all the momentum δ-functions in vertices are satisfied (e.g. in φ^3-theory instead of denoting legs going into the vertex as p_1, p_2, p_3, one denotes the third leg directly as $-p_1 - p_2$). The factor corresponding to a diagram is the product of factors corresponding to all external lines, internal lines and vertices of the diagram, multiplied by

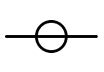
- an extra factor $\int \frac{d^4k}{(2\pi)^4}$ for each independent loop
- an overall momentum δ-function factor: $(2\pi)^4 \delta(P_f - P_i)$
  (P_f and P_f are sums of the ingoing and outgoing momenta respectively)
- an extra so-called combinatorial factor, to be discussed later
- some extra factors of (-1) related to fermionic lines

The reader may check, that in all the previous examples the results are obtained more directly with this formulation. The new rules are obtained from the previous one by performing all trivial integrations over the vertex δ-functions. To see this, let us ignore all factors corresponding to a diagram except of momentum integrations for internal legs and momentum δ-functions for vertices. Each integration corresponding to an internal line connecting two different vertices can now be depicted as omission of the corresponding internal line and merging two vertices into one vertex. Repeating this procedure over and over, one eventually gets rid of all internal lines connecting two different vertices. So finally one obtains a daisy-like diagram with only one vertex and some loopy internal lines going from this vertex and returning back. The number of these loops is the same, as the number of independent loops in the original diagram (this is due to the Euler's theorem $L = I - V + 1$ and the fact that at each step the numbers I and V decrease by one). The remaining integrals are the loop-integrals mentioned in the alternative formulation. The δ-function corresponding to the last vertex is the overall δ-function. (Convince yourself about the last two statements.)

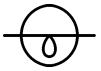
combinatorial factors

Beyond the tree level, a diagram may require the so-called combinatorial factor[16], which is usually the most cumbersome factor to evaluate. Therefore, it seems reasonable to start with some simple rules of thumb:

• If two vertices are connected by n different internal lines, the corresponding combinatorial factor is $1/n!$

 $\frac{1}{3!}$  $\frac{1}{2!2!}$

• If a line starts and ends in the same vertex, it contributes by $1/2$ to the combinatorial factor

 $\frac{1}{2}$  $\frac{1}{2}\frac{1}{2!}$

• If N permutations of n vertices do not change the diagram, the corresponding combinatorial factor is $1/N$ (note that if not all $n!$ permutations leave the diagram untouched, then $N \neq n!$)

 $\frac{1}{2}\frac{1}{3!}$  $\frac{1}{2^3}\frac{1}{3!}$

• The above list is not exhaustive, e.g. it does not allow to find the correct combinatorial factor in the following case

 $\frac{1}{8}$

A systematic, but less illustrative, prescription goes something like this: Let us assign a label to each end of every line in the diagram. The labeled diagram is characterized by sets of labels belonging to the common vertex, and by pairs of labels connected by a line. Permutation of labels would typically lead to a diagram labeled in a different way. Some permutations, however, lead to the identically labeled diagram. Find out the number N of all such permutations (leaving the labeled diagram unchanged). The combinatorial factor of the diagram is $1/N$.

Needless to say, this prescription is not that easy to follow practically. Fortunately, in simple cases (and one seldom needs to go much further) it can be reduced easily to the above rules of thumb. To provide the reader with a systematic procedure of generating all diagrams with right combinatorial factors, we will formulate one such method in the next paragraph. If found too clumsy, the paragraph can be skipped safely.

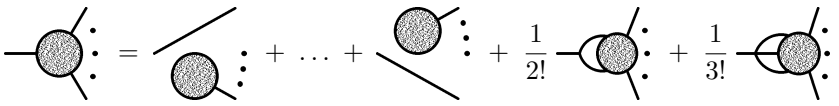
[16] Why such a name: the diagrams represent specific terms of perturbative expansion, the number of terms corresponding to a given diagram is given by some combinatorics.

a systematic way of drawing the diagrams with correct combinatorial factors

Let us consider diagrams with l external legs. We will represent the sum of all such diagrams (including diagrams with no vertices at all, diagrams containing vacuum bubbles and diagrams containing corrections on external legs) by a shaded blob with l legs.

Now let us focus on one of the external legs. What is it connected to? It can be connected either directly to one of the other external legs, or it goes to some m-leg vertex. In the former case, the sum of all diagrams equals to one leg with no vertices at all and the sum of all diagrams with l–2 external legs (a simple line disconnected to a (l–2)-legged shaded blob). In the latter case, the sum of all diagrams contains this leg going to the said vertex and the sum of all diagrams with l+m–2 external legs, m–1 coming from the said vertex and l–1 being true external legs (the vertex connected to a l+m–2 shaded blob). If the external leg can go to different vertices, all of them has to be taken into account.

The combinatorial factors come as follows: If the vertex contains n_1, n_2, ... non-external legs corresponding to the same field, then the combinatorial factor is $1/(n_1!\ n_2!\ \ldots)$. The result is the known as the Dyson-Schwinger equation. For combined φ^3- and φ^4-theories it reads

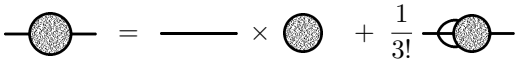


To get all diagrams up to a given order with correct combinatorial factors, the DS equation is used in an iterative way: one starts the equation itself, then one takes any leg and applies the equation to it, then the same is repeated with some other leg etc., until one reaches

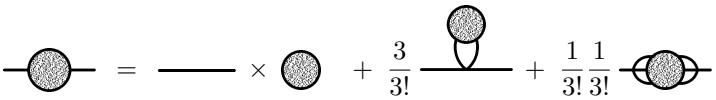
(the structure one is interested in) $\times$  $+\ldots$

with ellipsis standing for diagrams with disconnected external legs + higher orders.

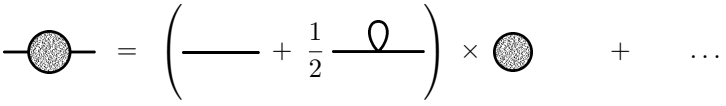
Let us illustrate the procedure by the diagram with two external legs within the φ^4-theory. The starting point is the DS equation for 2 external legs



Now the DS equation is applied to some other leg, say to the 2nd external leg

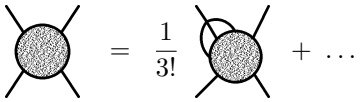


If we are interested only in the 1st order (in the number of vertices), then the last term is already of higher order, and the second term is again processed by the DS equation, to finally give

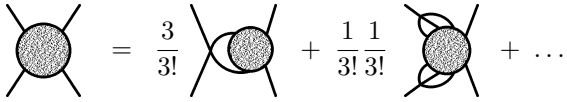


The factor in front of the second diagram in the brackets is the correct combinatorial factor for this diagram. (As an exercise the reader may try to go one order higher.)

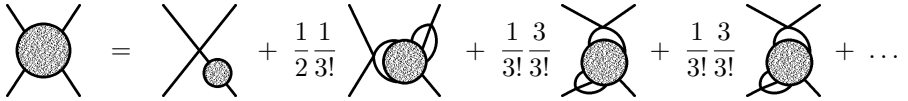
As another example let us consider the $AB \to 12$ scattering within the φ^4-theory. Again, the starting point is the DS equation



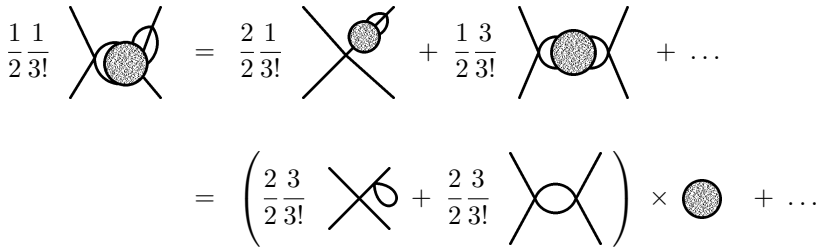
where the ellipsis stands for terms with disconnected external legs. The DS equation is now applied to some other leg, say the external leg B



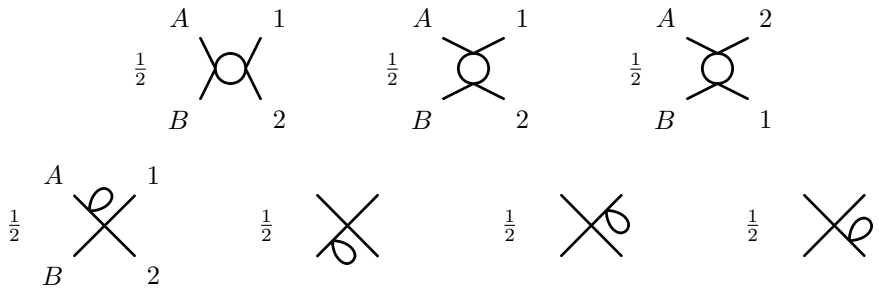
The RHS enjoys yet another DS equation, now say on the external leg 1, to give



The first two (last two) terms on the RHS come from the first (second) diagram on the RHS above, terms with more than two vertices are included into ellipsis. The first diagram is now treated with the help of the previous result for the 2-leg diagrams, the other three are treated all in the same way, which we will demonstrate only for the second diagram: we use the DS equation once more, now say for the external leg 2



Putting the pieces together, one finally obtains the one-loop diagrams with the correct combinatorial factors



1.1.3 Scattering amplitude

The definition of the scattering amplitude M_{fi} is quite simple:

$$\text{the sum of Feynman diagrams} = iM_{fi}\,(2\pi)^4\,\delta^{(4)}\,(P_f - P_i)$$

where P_i and P_f are the overall initial and final momentum respectively. By the sum of the diagrams, the sum of the corresponding factors is meant, of course.

Examples (to be checked by the reader):

- **φ^3-theory,** $AB \to 12$ **scattering**

$$\text{tree-level} \quad M_{fi} = -\frac{g^2}{(p_A+p_B)^2-m^2} - \frac{g^2}{(p_A-p_1)^2-m^2} - \frac{g^2}{(p_A-p_2)^2-m^2}$$

1-loop-level the result is intricate and not that illuminating
but the reader is encouraged to work out some loop diagrams

- **φ^4-theory,** $AB \to 12$ **scattering**

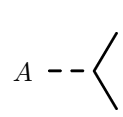
$$\text{tree-level} \quad M_{fi} = -g$$

$$\text{1-loop-level} \quad M_{fi} = -\tfrac{1}{2}g^2\left[I\,(p_A+p_B) + I\,(p_A-p_1) + I\,(p_A-p_2)\right]$$

$$I\,(p) = i\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2}\,\frac{1}{(p-k)^2-m^2}$$

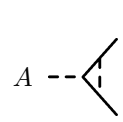
- **$\varphi^2\Phi$-theory**[17], $A \to 12$ **decay**

tree-level



$$M_{fi} = -g$$

1-loop-level



$$M_{fi} = -g^3 J\,(p_1,p_2)$$

$$J\,(p_1,p_2) = i\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-M^2}\,\frac{1}{(p_1+k)^2-m^2}\,\frac{1}{(p_2-k)^2-m^2}$$

[17] A theory for two different fields φ and Φ, defined by the Lagrangian

$$\mathcal{L}\,[\varphi,\Phi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 - \frac{g}{2}\varphi^2\Phi$$

1.1.4 Cross-sections and decay rates

The cross-section for a scattering $AB \to 12\ldots n$ is given by

$$d\sigma = (2\pi)^4 \delta^4 (P_f - P_i) \frac{1}{4\sqrt{(p_A.p_B)^2 - m_A^2 m_B^2}} |M_{fi}|^2 \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

while the analogous quantity for a decay $A \to 12\ldots n$ is

$$d\Gamma = (2\pi)^4 \delta^4 (P_f - P_i) \frac{1}{2E_A} |M_{fi}|^2 \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

Because of the δ-function present in these formulae, one can relatively easily perform four integrations on the RHS. For the quite important case of $n = 2$, i.e. for $AB \to 12$ and $A \to 12$, the result after such integrations is in the CMS[18]

$$d\sigma_{\mathrm{CMS}} = \frac{1}{64\pi^2} \frac{|\vec{p}_1|}{|\vec{p}_A|} \frac{1}{(p_A + p_B)^2} |M_{fi}|^2 \, d\Omega_1$$

$$d\Gamma_{\mathrm{CMS}} = \frac{1}{32\pi^2} \frac{|\vec{p}_1|}{m_A^2} |M_{fi}|^2 \, d\Omega_1$$

Examples:

- **φ^4-theory,** $AB \to 12$ **scattering,** tree level

  $$d\sigma_{\mathrm{CMS}} = \frac{1}{64\pi^2} \frac{1}{s} g^2 d\Omega \qquad s = (p_A + p_B)^2$$

  In this case the differential cross-section does not depend on angles, so one can immediately write down the total cross-section $\sigma_{\mathrm{CMS}} = g^2/16\pi s$.

- **φ^3-theory,** $AB \to 12$ **scattering,** tree level

  $$d\sigma_{\mathrm{CMS}} = \frac{g^4}{64\pi^2 s} \left(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2} \right)^2 d\Omega \qquad \begin{array}{l} t = (p_A - p_1)^2 \\ u = (p_A - p_2)^2 \end{array}$$

  where s, t, u are the frequently used so-called Mandelstam variables.

Exercises:

- $AB \to 12$ scattering at the tree level, within "the φ^4-theory with derivatives", i.e. the theory of scalar fields with the non-quadratic part of the Lagrangian being $\mathcal{L}_{\mathrm{int}} = -\frac{g}{4} \varphi^2 \partial_\mu \varphi \partial^\mu \varphi$.
- $\Phi \to \varphi\varphi$ decay rate, $\varphi\varphi \to \varphi\varphi$, $\varphi\Phi \to \varphi\Phi$ and $\varphi\varphi \to \Phi\Phi$ cross-sections at the tree level, for the $\varphi^2\Phi$-theory defined in the footnote on the page 17.

[18] Once the result is known in the CMS, one can rewrite it into any other system, but this is not trivial, since experimentally natural quantities like angles are not Lorentz covariant. It is therefore useful to have explicit formulae for commonly used systems, e.g. for the so-called target system (the rest frame of the particle B)

$$d\sigma_{\mathrm{TS}} = \frac{1}{64\pi^2} \frac{|\vec{p}_1|}{|\vec{p}_A|} \frac{1}{m_B} \frac{1}{(E_A + m_B) |\vec{p}_1| - E_A |\vec{p}_A| \cos\vartheta} |M_{fi}|^2 \, d\Omega_1$$

where all quantities $(\vec{p}, E, \vartheta, \Omega)$ are understood in the target frame.

1.2 Many-Body Quantum Mechanics

The main characters of QFT are quantum fields or perhaps creation and annihilation operators (since the quantum fields are some specific linear combinations of the creation and annihilation operators). In most of the available textbooks on QFT, the creation and annihilation operators are introduced in the process of the so-called canonical quantization[19]. This, however, is not the most natural way. In opinion of the present author, it may be even bewildering, as it may distort the student's picture of relative importance of basic ingredients of QFT (e.g. by overemphasizing the role of the canonical quantization). The aim of this second introduction is to present a more natural definition of the creation and annihilation operators, and to demonstrate their main virtues.

1.2.1 Fock space, creation and annihilation operators

Fock space

1-particle system
the states constitute a Hilbert space $\mathcal{H}^1$ with an orthonormal basis $|i\rangle$, $i \in N$

2-particle system[20]
the states constitute the Hilbert space $\mathcal{H}^2$ or $\mathcal{H}^2_{\rm B}$ or $\mathcal{H}^2_{\rm F}$, with the basis $|i,j\rangle$

non-identical particles	$\mathcal{H}^2 = \mathcal{H}^1 \otimes \mathcal{H}^1$	$\|i,j\rangle = \|i\rangle \otimes \|j\rangle$
identical bosons	$\mathcal{H}^2_{\rm B} \subset \mathcal{H}^1 \otimes \mathcal{H}^1$	$\|i,j\rangle = \frac{1}{\sqrt{2}} (\|i\rangle \otimes \|j\rangle + \|j\rangle \otimes \|i\rangle)$
identical fermions	$\mathcal{H}^2_{\rm F} \subset \mathcal{H}^1 \otimes \mathcal{H}^1$	$\|i,j\rangle = \frac{1}{\sqrt{2}} (\|i\rangle \otimes \|j\rangle - \|j\rangle \otimes \|i\rangle)$

n-particle system (identical particles)
the Hilbert space is either $\mathcal{H}^n_{\rm B}$ or $\mathcal{H}^n_{\rm F} \subset \underbrace{\mathcal{H}^1 \otimes \ldots \otimes \mathcal{H}^1}_{n}$, with the basis

$$|i,j,\ldots,k\rangle = \frac{1}{\sqrt{n!}} \sum_{\text{permutations}} (\pm 1)^p \underbrace{|i\rangle \otimes |j\rangle \otimes \ldots \otimes |k\rangle}_{n}$$

where p is the parity of the permutation, the $\begin{matrix}\text{upper}\\ \text{lower}\end{matrix}$ sign applies to $\begin{matrix}\text{bosons}\\ \text{fermions}\end{matrix}$

0-particle system
1-dimensional Hilbert space $\mathcal{H}^0$ with the basis vector $|0\rangle$ (no particles, vacuum)

Fock space
direct sum of the bosonic or fermionic n-particle spaces

$$\mathcal{H}_{\rm B} = \bigoplus_{n=0}^{\infty} \mathcal{H}^n_{\rm B} \qquad\qquad \mathcal{H}_{\rm F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^n_{\rm F}$$

[19] There are exceptions. In the Weinberg's book the creation and annihilation operators are introduced exactly in the spirit we are going to adopt in this section. The same philosophy is to be found in some books on many-particle quantum mechanics. On the other hand, some QFT textbooks avoid the creation and annihilation operators completely, sticking exclusively to the path integral formalism.

[20] This is the keystone of the whole structure. Once it is really understood, the rest follows smoothly. To achieve a solid grasp of the point, the reader may wish to consult the couple of remarks following the definition of the Fock space.

Remark: *Let us recall that for two linear spaces U (basis e_i, dimension m) and V (basis f_j, dimension n), the direct sum and product are linear spaces $U \oplus V$ (basis generated by both e_i and f_j, dimension $m+n$) and $U \otimes V$ (basis generated by ordered pairs (e_i, f_j), dimension $m.n$).*

Remark: *The fact that the Hilbert space of a system of two non-identical particles is the direct product of the 1-particle Hilbert spaces may come as not completely obvious. If so, it is perhaps a good idea to start from the question what exactly the 2-particle system is (provided that we know already what the 1-particle system is). The answer within the quantum physics is not that straightforward as in the classical physics, simply because we cannot count the quantum particles directly, e.g. by pointing the index finger and saying one, two. Still, the answer is not too complicated even in the quantum physics. It is natural to think of a quantum system as being 2-particle iff*
a) it contains states with sharp quantum numbers (i.e. eigenvalues of a complete system of mutually commuting operators) of both 1-particle systems, and this holds for all combinations of values of these quantum numbers
b) such states constitute a complete set of states
This, if considered carefully, is just the definition of the direct product.

Remark: *A triviality which, if not explicitly recognized, can mix up one's mind: $\mathcal{H}^1 \cap \mathcal{H}^2 = \emptyset$, i.e. the 2-particle Hilbert space contains no vectors corresponding to states with just one particle, and vice versa.*

Remark: *The fact that multiparticle states of identical particles are represented by either completely symmetric or completely antisymmetric vectors should be familiar from the basic QM course. The former case is called bosonic, the latter fermionic. In all formulae we will try, in accord with the common habit, to treat these two possibilities simultaneously, using symbols like $\pm$ and $\mp$, where the upper and lower signs apply to bosons and fermions respectively.*

Remark: *As to the basis vectors, our notation is not the only possible one. Another widely used convention (the so-called occupation number representation) denotes the basis vectors as $|n_1, n_2, \ldots\rangle$, where n_i is the number of particles in the i-th 1-particle state. So e.g. $|2,2,2,4,4\rangle \Leftrightarrow |0,3,0,2,0,0,0,\ldots\rangle$, where the LHS is in our original notation while the RHS is in the occupation number representation. The main drawback of the original notation is that it is not unique, e.g. $|1,2,3\rangle$ and $\pm|1,3,2\rangle$ denotes the same vector. One should be therefore careful when summing over all basis states. The main drawback of the occupation number representation is typographical: one cannot write any basis vector without the use of ellipsis, and even this may sometimes become unbearable (try e.g. to write $|49,87,642\rangle$ in the occupation number representation).*

Remark: *The basis vectors $|i,j,\ldots,k\rangle$ or $|n_1,n_2,\ldots\rangle$ are not all normalized to unity (they are, but only if all $i,j,\ldots,k$ are mutually different, i.e. if none of n_i exceeds 1). If some of the $i,j,\ldots,k$ are equal, i.e. if at least one $n_i > 1$, then the norm of the fermionic state is automatically zero (this is the Pauli exclusion principle), while the norm of the bosonic state is $\sqrt{n_1!\, n_2!\, \ldots}$. Prove this.*

Remark: *A triviality which, if not explicitly recognized, can mix up one's mind: the vacuum $|0\rangle$ is a unit vector which has nothing to do with the zero vector 0.*

creation and annihilation operators

Let $|i\rangle$ $(i = 1, 2, \ldots)$ be an orthonormal basis of a 1-particle Hilbert space, and $|0\rangle$, $|i\rangle$, $|i,j\rangle$, $|i,j,k\rangle$, $\ldots$ $(i \leq j \leq k \leq \ldots)$ an orthogonal basis of the Fock space. The creation and annihilation operators are defined as follows

creation operator a_i^+
is a linear operator, which maps the n-particle basis vector to the $(n+1)$-particle vector by adding one particle in the i-th state (the particle is added at the first position in the resulting vector; for bosons this rule does not matter, for fermions it determines the sign)

$$a_i^+ |0\rangle = |i\rangle \qquad a_i^+ |j\rangle = |i,j\rangle \qquad a_i^+ |j,k,\ldots\rangle = |i,j,k,\ldots\rangle$$

annihilation operator a_i
is a linear operator, which maps the n-particle basis vector to the $(n-1)$-particle vector by removing one particle in the i-th state. The particle is removed from the first position of the original vector, and if it is not there, the original vector must be reshuffled (for bosons this rule does not matter, for fermions it determines the sign). If the original vector contains more than one particle in the i-th state, the whole procedure is performed with each of them and the results are summed up. If the original vector does not contain a particle in the i-th state, the result is the zero vector.

$$a_i |0\rangle = 0 \qquad a_i |j\rangle = \delta_{ij} |0\rangle$$
$$a_i |j,k,l\ldots\rangle = \delta_{ij} |k,l,\ldots\rangle \pm \delta_{ik} |j,l,\ldots\rangle + \delta_{il} |j,k,\ldots\rangle \pm \ldots$$

Both creation and annihilation operators are linear and they are defined on the basis vectors. Consequently they are defined for any vector.

Remark: *In the occupation number representation, the definitions read*

$$\begin{aligned}
&\textit{bosons} \quad && a_i^+ |n_1,\ldots,n_i,\ldots\rangle = |n_1,\ldots,n_i+1,\ldots\rangle \\
& && a_i |n_1,\ldots,n_i,\ldots\rangle = n_i |n_1,\ldots,n_i-1,\ldots\rangle \\
&\textit{fermions} \quad && a_i^+ |n_1,\ldots,n_i=0,\ldots\rangle = (-1)^{p_i} |n_1,\ldots,n_i=1,\ldots\rangle \\
& && a_i^+ |n_1,\ldots,n_i=1,\ldots\rangle = 0 \\
& && a_i |n_1,\ldots,n_i=0,\ldots\rangle = 0 \\
& && a_i |n_1,\ldots,n_i=1,\ldots\rangle = (-1)^{p_i} |n_1,\ldots,n_i=0,\ldots\rangle \\
& && p_i = \textstyle\sum_{k=1}^{i-1} n_k
\end{aligned}$$

Creation and annihilation operators are very useful, because
- they enable the most natural description of processes in which the number of particles is not conserved, i.e. in which particles are created and/or destroyed
- any linear operator can be expressed in terms of the creation and annihilation operators, namely as a sum of products of these operators
- there is a standard and routine method of how to calculate matrix elements of operators expressed in terms of the creation and annihilation operators.

In view of how frequent the processes of particle creation and annihilation are (decays and inelastic scatterings in the atomic, nuclear, subnuclear and solid state physics), the first point is evidently very important. And in view of how often the QM calculations are just the calculations of various matrix elements of linear operators, the other two points are clearly also very important.

key attributes of the creation and annihilation operators

Perhaps the three most important are[21]

- $a_i^+ = a_i^\dagger$ i.e. a_i^+ and a_i are Hermitian conjugated
- $a_i^+ a_i$ (no summation) is the operator of number of particles in the i-th state
- $\left[a_i, a_j^+\right]_\mp = \delta_{ij} \qquad [a_i, a_j]_\mp = \left[a_i^+, a_j^+\right]_\mp = 0 \qquad$ where $[x,y]_\mp = xy \mp yx$

The proof is an easy and very useful exercise, recommended to anybody who wants to become quickly accustomed to elementary manipulations with the a_i^+, a_i operators. The following sketch of the proof is therefore intended only as a check of reader's own work (the proof is performed in the occupation number formalism, which is more convenient for this purpose).

- Hermitian conjugation

$$\begin{aligned}
&\langle \ldots n_i' \ldots |\, a_i \,| \ldots n_i \ldots \rangle = \langle \ldots n_i' .. | . n_i - 1 \ldots \rangle \, n_i = \langle \ldots | \ldots \rangle \, (n_i - 1)! \; n_i \; \delta_{n_i', n_i - 1} \\
&\langle \ldots n_i \ldots |\, a_i^+ \,| \ldots n_i' \ldots \rangle = \langle \ldots n_i .. | . n_i' + 1 \ldots \rangle \quad = \langle \ldots | \ldots \rangle \, n_i! \; \delta_{n_i, n_i' + 1}
\end{aligned}$$

where for bosons $n_i, n_i' \in N$ and for fermions $n_i, n_i' \in \{0, 1\}$

- particle number operator

bosons
$$a_i^+ a_i \,| \ldots n_i \ldots \rangle = a_i^+ n_i \,| \ldots n_i - 1 \ldots \rangle = n_i a_i^+ \,| \ldots n_i - 1 \ldots \rangle = n_i \,| \ldots n_i \ldots \rangle$$

fermions
$$\begin{aligned}
&a_i^+ a_i \,| \ldots \, 0 \, \ldots \rangle = 0 \\
&a_i^+ a_i \,| \ldots \, 1 \, \ldots \rangle = a_i^+ \, (-1)^{p_i} \,| \ldots 0 \ldots \rangle = (-1)^{2p_i} \,| \ldots 1 \ldots \rangle
\end{aligned}$$

- (anti)commutation relation

bosons
$$\begin{aligned}
\left[a_i, a_i^+\right] | \ldots n_i \ldots \rangle &= a_i \,| \ldots n_i + 1 \ldots \rangle - n_i a_i^+ \,| \ldots n_i - 1 \ldots \rangle \\
&= (n_i + 1) \,| \ldots n_i \ldots \rangle - n_i \,| \ldots n_i \ldots \rangle = | \ldots n_i \ldots \rangle \\
\left[a_i, a_j^+\right] | \ldots n_i \ldots n_j \ldots \rangle &= a_i \,| \ldots n_i \ldots n_j + 1 \ldots \rangle - a_j^+ \,| \ldots n_i - 1 \ldots n_j \ldots \rangle \\
&= | \ldots n_i - 1 \ldots n_j + 1 \ldots \rangle - | \ldots n_i - 1 \ldots n_j + 1 \ldots \rangle = 0
\end{aligned}$$

fermions
$$\begin{aligned}
&\left\{a_i, a_i^+\right\} | \ldots 1 \ldots \rangle = 0 + (-1)^{p_i} \, a_i^+ \,| \ldots 0 \ldots \rangle = (-1)^{2p_i} \,| \ldots 1 \ldots \rangle = | \ldots 1 \ldots \rangle \\
&\left\{a_i, a_i^+\right\} | \ldots 0 \ldots \rangle = (-1)^{p_i} \, a_i \,| \ldots 1 \ldots \rangle + 0 = (-1)^{2p_i} \,| \ldots 0 \ldots \rangle = | \ldots 0 \ldots \rangle \\
&\left\{a_i, a_j^+\right\} | \ldots 0 \ldots 0 \ldots \rangle = (-1)^{p_j} \, a_i \,| \ldots 0 \ldots 1 \ldots \rangle + 0 = 0 \\
&\left\{a_i, a_j^+\right\} | \ldots 0 \ldots 1 \ldots \rangle = 0 \\
&\left\{a_i, a_j^+\right\} | \ldots 1 \ldots 0 \ldots \rangle = (-1)^{p_i + p_j} \,| \ldots 0 \ldots 1 \ldots \rangle + (-1)^{p_i + p_j - 1} \,| \ldots 0 \ldots 1 \ldots \rangle = 0 \\
&\left\{a_i, a_j^+\right\} | \ldots 1 \ldots 1 \ldots \rangle = 0 + (-1)^{p_i} \, a_j^+ \,| \ldots 0 \ldots 1 \ldots \rangle = 0
\end{aligned}$$

The other (anti)commutation relations are treated in the same way.

[21] Note that a_i^+ and a_i operators could be (and often are) introduced in the reversed order. In that case, the (anti)commutation relations are postulated and the Fock space is constructed afterwards for a_i^+ and a_i to have something to live in. It is perhaps just a matter of taste, but the present author strongly prefers the "more natural logic" of this section. Later in these lectures, however, we will encounter also the reversed logic of the second quantization.

The (anti)commutation relations comprise almost everything we would need as to the creation and annihilation operators. Still, there are some additional useful relations, like the relation between different sets of the creation and annihilation operators.

Let $|\alpha\rangle$ ($\alpha \in N$) be an orthonormal basis of the 1-particle Hilbert space, different from the original basis $|i\rangle$. Starting from this new basis, one can define the new set of creation and annihilation operators a_α^+ and a_α. The relation between these operators and the original ones reads

$$a_\alpha^+ = \sum_i \langle i|\alpha\rangle \, a_i^+ \qquad\qquad a_\alpha = \sum_i \langle \alpha|i\rangle \, a_i$$

This follows directly from the relation $|\alpha\rangle = |i\rangle \langle i|\alpha\rangle$ (with the Einstein summation convention understood) and from the definition of the creation and annihilation operators (convince yourself that this, indeed, is true[22]).

These relations represent a concise dictionary for important translations from one 1-particle basis to another one (e.g. from the x-representation to the p-representation and vice versa.)

Remark: *So one can have several different sets of creation and annihilation operators defined by different choices of basis in* 1*-particle Hilbert space. On top of that, one can encounter yet another — completely different — set of creation and annihilation operators.*
The point is that the basis $|i, j, \ldots\rangle$ *(or* $|n_1, n_2, \ldots\rangle$ *in the occupation number formalism), which arises from a particular basis* $|i\rangle$ *of the* 1*-particle Hilbert space, is perhaps the most natural, but not the only reasonable, basis in the Fock space. Actually, any complete set of (physically relevant) commuting operators defines some (relevant) basis. (From this point of view, the basis* $|n_1, n_2, \ldots\rangle$ *is just the basis of eigenvectors of the occupation number operators.)*
If the eigenvalues of a complete system of commuting operators are discrete and bounded from below, then one can label both eigenvalues and eigenvectors by natural numbers. In such a case, the basis defined by the considered system of operators looks like $|N_1, N_2, \ldots\rangle$*, and for a basis of this type we can define the so-called raising and lowering operators, just as we have defined the creation and annihilation operators:* A_i^+ *(*A_i*) raises (lowers) the quantum number* N_i *by one.*
Of a special interest are the cases when the Hamiltonian can be written as a sum of two terms, one of which has eigenvectors $|N_1, N_2, \ldots\rangle$ *and the second one can be understood as a small perturbation. If so, the system formally looks like an almost ideal gas made of a new type of particles (created by the* A_i^+ *operators from the state in which all* N_i *vanish). These formal particles are not to be mistaken for the original particles, which the system is built from.*
It may come as a kind of surprise that such formal particles do appear frequently in many-body systems. They are called elementary excitations, and they come in great variety (phonons, plasmons, magnons, etc.). Their relation to the original particles is more or less known as the result of either detailed calculations, or an educated guess, or some combination of the two. The description of the system is, as a rule, much simpler in terms of the elementary excitations than in terms of the original particles. This explains the wide use of the elementary excitations language by both theorists and experimentalists.

[22] One should build the Fock space basis starting from the basis $|\alpha\rangle$ of the one-particle states, then to define the corresponding creation and annihilation operators and then to check the relation between the original and the new operators.

Remark: *The reader is perhaps already familiar with operators a^+ and a, satisfying the above commutation relations. Such operators are usualy introduced for the LHO (linear harmonic oscillator) in a basic QM course*[23]*. What is the relation between the LHO a^+ and a operators and the ones discussed here?*
The famous LHO a^+ and a operators can be viewed as the creation and annihilation operators of the elementary excitations for the (extreme) example of a many-body system presented by one LHO. The point is that a system with one particle that can be in any of infinite number of states, is formally equivalent to the ideal gas of arbitrary number of formal particles, all of which can be, however, in just one state.
Do the LHO operators a^+ and a play any role in QFT? Yes, they do, but only an auxiliary role and only in one particular development of QFT, namely in the canonical quantization of classical fields. But since this is still perhaps the most common development of the theory, the role of the LHO is easy to be overestimated. Anyway, as for the present section the reader may well forget about the LHO.

Remark: *An ideal gas of formal particles, which arises more or less naturally in discussion of LHO, is even more appealing in case of coupled harmonic oscilators. And this is indeed a very important case, due to the famous miracle of systems in the vicinity of their stable equilibriums: any such system is well approximated by the system of coupled harmonic oscillators which, in turn, is equivalent to the system of* decoupled *harmonic oscillators.*
Stationary states of the system of the independent harmonic oscillators are characterized by the sequence $(N_1, N_2, \ldots)$ where N_i defines the energy level of the i-th oscillator Energies of individual oscillators are $\hbar\omega_n(N_n + 1/2)$ where $\omega_n = \sqrt{K_n/M}$. Energy of the system is $\sum_i \hbar\omega_n(N_n + 1/2)$. Now let us imagine a system of free particles, each of them having energy eigenstates labeled by n with eigenvalues $\hbar\omega_n$. If there are N_n particles in the n-th state, the energy of the system will be $\sum_n \hbar\omega_n N_n$. This is equivalent (up to a constant) to the Hamiltonian of independent oscillators. It is common habit to describe a system of independent harmonic oscillators in terms of the equivalent system of free formal particles. These formal particles are called phonons.
Phonons are would-be particles widely used for the formal description of real particles (harmonic oscillators). These formal particles may look like real ones, especially in systems with translational symmetry. In such systems decoupling of oscilators is provided by the Fourier transformation, which brings (quasi)momentum in the game. In that case phonons behave like having well defined energy and (quasi)momentum. Nevertheles, phonon is not a kind of particle. Strictly speaking, it is just a word.

[23] Recall $a^+ = \hat{x}\sqrt{m\omega/2\hbar} - i\hat{p}/\sqrt{2\hbar m\omega}$, $a = \hat{x}\sqrt{m\omega/2\hbar} + i\hat{p}/\sqrt{2\hbar m\omega}$. The canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$ implies for a^+ and a the commutation relation of the creation and annihilation operators. Moreover, the eigenvalues of the operator $N = a^+a$ are natural numbers (this follows from the commutation relation).
Note that the definition of a^+ and a (together with the above implications) applies to any 1-particle system, not only to the LHO. What makes the LHO special in this respect is the Hamiltonian, which is very simple in terms of a^+, a. These operators are useful also for the systems "close to LHO", where the difference can be treated as a small perturbation.

1.2.2 Important operators expressed in terms of a_i^+, a_i

As already announced, any linear operator in the Fock space can be written as a polynomial (perhaps infinite) in creation and annihilation operators. However interesting this general statement may sound, the particular examples are even more interesting and very important in practice. We will therefore start with the examples and return to the general statement only later on.

Hamiltonian of a system of non-interacting particles

Let us consider non-interacting particles (ideal gas) in an external classical field with a potential energy $U(x)$. The most suitable choice of basis in the 1-particle Hilbert space is the set of eigenstates of the 1-particle Hamiltonian $\hat{p}^2/2m + U(x)$, i.e. the states $|i\rangle$ satisfying

$$\left(\frac{1}{2m}\hat{p}^2 + U(x)\right)|i\rangle = E_i\,|i\rangle$$

By this choice, the standard basis of the whole Fock space is determined, namely $|0\rangle$, $|i\rangle$, $|i,j\rangle$, $|i,j,k\rangle$, etc. And since the particles do not interact, each of these basis states has a sharp value of energy, namely 0, E_i, E_i+E_j, $E_i+E_j+E_k$, etc., respectively. The Hamiltonian of the system with any number of particles is the linear operator with these eigenvectors and these eigenvalues. It is very easy to guess such an operator, namely $H_0 = \sum_i E_i \hat{n}_i$, where $\hat{n}_i$ is the operator of number of particles in the i-th state. And since we know how to express $\hat{n}_i$ in terms of the creation and annihilation operators, we are done

$$H_0 = \sum_i E_i\; a_i^+ a_i$$

- If, for any reason, we would need to express H_0 in terms of another set of creation and annihilation operators $a_\alpha^+ = \sum_i \langle i|\alpha\rangle\, a_i^+$ and $a_\alpha = \sum_i \langle\alpha|i\rangle\, a_i$, it is straightforward to do so: $H_0 = \sum_{\alpha,\beta} E_{\alpha\beta} a_\alpha^+ a_\beta$ where $E_{\alpha\beta} = \sum_i E_i \langle\alpha|i\rangle\,\langle i|\beta\rangle$.
- If one has a continuous quantum number q, rather than the discrete i, then the sum $\sum_i$ is replaced by the integral $\int dq$: $H_0 = \int dq\; E(q)\, a_q^+ a_q$. Another change is that any Kronecker δ_{ij} is replaced by the Dirac $\delta(q-q')$.

Free particles. 1-particle states labeled by the momentum $\vec{p}$, with $E(\vec{p}) = \frac{p^2}{2m}$

$$H_0 = \int d^3p\; \frac{p^2}{2m}\; a_{\vec{p}}^+ a_{\vec{p}}$$

Periodic external field (the 0-th approximation for electrons in solids). Bloch theorem: 1-particle states are labeled by the level n, the quasi-momentum $\vec{k}$, and the spin σ. The energy $\varepsilon_n(\vec{k})$ depends on details of the periodic field

$$H_0 = \sum_{n,\sigma} \int d^3k\; \varepsilon_n(\vec{k})\; a^+_{n,\vec{k},\sigma} a_{n,\vec{k},\sigma}$$

Spherically symmetric external field (the 0-th approximation for electrons in atoms and nucleons in nuclei). 1-particle states are labeled by the quantum numbers n, l, m, σ. The energy $E_{n,l}$ depends on details of the field

$$H_0 = \sum_{n,l,m,\sigma} E_{n,l}\; a^+_{n,l,m,\sigma} a_{n,l,m,\sigma}$$

Remark: *The ideal gas approximation is very popular for electrons in atoms, molecules and solids. At the first sight, however, it looks like a rather poor approximation. The dominant (Coulomb) interaction between electrons is enormous at atomic scale and cannot be neglected in any decent approach.*
But there is no mystery involved, the ideal gas approximation used for electrons does not neglect the Coulomb interaction. The point is that the external field for the electron ideal gas contains not only the Coulomb field of positively charged nuclei, but also some kind of mean field of all negatively charged electrons. This mean field is usually given by the Hartree-Fock approximation. The corner-stone of this approximation is a restriction on electron states taken into consideration: only the direct products of single electron states are accounted for. In this restricted set of states one looks for what in some sense is the best approximation to the stationary states. This leads to the specific integro-differential equation for the 1*-electron states and corresponding energies, which is then solved iteratively*[24]*. The creation and annihilation operators for these Hartree-Fock states and the corresponding energies then enter the electron ideal gas Hamiltonian.*

Remark: *The ground state of a fermionic system in the Hartree-Fock approximation (the ideal gas approximation with* 1*-particle states and energies given by the Hartree-Fock equation) is quite simple: all the* 1*-particle states with energies below some boundary energy, the so-called Fermi energy ε_F, are occupied, while all the states with energies above ε_F are free. The Fermi energy depends, of course, on the number of particles in the system.*
In solids, the 1*-particle Hartree-Fock states are characterized by $(n, \vec{k}, \sigma)$ (level, quasi-momentum, spin). The* 1*-particle n-th level Hartree-Fock energy is, as a rule, an ascending function of k^2 in any direction of $\vec{k}$. In any direction, therefore, there exists the level n and the vector $\vec{k}_F(\varphi, \vartheta)$ for which $\varepsilon_n(\vec{k}_F) = \varepsilon_F$. The endpoints of vectors $\vec{k}_F(\varphi, \vartheta)$ form a surface, called the Fermi surface. In the many-body ground state, the* 1*-particle states beneath (above) the Fermi surface are occupied (free).*
It turns out that for a great variety of phenomena in solids, only the low excited states of the electron system are involved. They differ from the ground state by having a few 1*-particle states above the Fermi surface occupied. The particular form of the Fermi surface therefore determines many macroscopic properties of the material under consideration. For this reason the knowledge of the Fermi surface is very important in the solid state physics.*

Remark: *The ideal gas of fermions is frequently treated by means of a famous formal trick known as the electron-hole formalism. The ground state of the N fermion ideal gas is called the Fermi vacuum, and denoted by $|0_F\rangle$. For $i \leq N$ one defines new operators $b_i^+ = a_i$ and $b_i = a_i^+$. The original a_i^+ and a_i operators are taken into account only for $i > N$.*
Both a- and b-operators satisfy the commutation relations, and both b_i $(i \leq N)$ and a_i $(i > N)$ annihilate the Fermi vacuum (indeed, $b_i |0_F\rangle = 0$ because of anti-symmetry of fermion states, i.e. because of the Pauli exclusive principle). So, formally we have two types of particles, the holes and the new electrons, created from the Fermi vacuum by b_i^+ and a_i^+ respectively. The Hamiltonian reads $H_0 = \sum_{i \leq N} E_i b_i b_i^+ + \sum_{i>N} E_i a_i^+ a_i = \sum_{i \leq N} E_i - \sum_{i \leq N} E_i b_i^+ b_i + \sum_{i>N} E_i a_i^+ a_i$. The popular interpretation of the minus sign: the holes have negative energy.

[24] For details consult any reasonable textbook on QM or solid state physics.

Hamiltonian of a system of particles with the pair interaction

Perhaps the most important interaction to be added to the previous case of the ideal gas is the pair interaction, i.e. an interaction characterized by a potential energy of pairs of particles (most of applications in the solid state, atomic and nuclear physics involve such an interaction). In this case, the most suitable choice of basis in the 1-particle Hilbert space is the x-representation $|\vec{x}\rangle$, since the 2-particle states $|\vec{x},\vec{y}\rangle$ have a sharp value of the pair potential energy $V(\vec{x},\vec{y})$.

Due to the fact that we are dealing with the pair interaction, the 3-particle state $|\vec{x}_1,\vec{x}_2,\vec{x}_3\rangle$ does also have the sharp value of the potential energy, namely $V(\vec{x}_1,\vec{x}_2)+V(\vec{x}_1,\vec{x}_3)+V(\vec{x}_2,\vec{x}_3)$, and the same holds for other multiparticle states (this, in fact, is the definition of the pair interaction).

What is the potential energy of the state with $n(\vec{x}_i)$ particles at the position $\vec{x}_i$, where $i=1,2,\ldots$? The number of pairs contributing by $V(\vec{x}_i,\vec{x}_j)$ is $\frac{1}{2}n_{\vec{x}_i}n_{\vec{x}_j}$ for $i\neq j$, by which we understand also $\vec{x}_i\neq\vec{x}_j$ (the $\frac{1}{2}$ is there to avoid double-counting). For $i=j$ there is a subtlety involved. One has to include the potential energy of a particle with all other particles sharing the same position, but not with itself (a particle with itself does not constitute a pair). The number of pairs contributing by $V(\vec{x}_i,\vec{x}_i)$ is therefore $\frac{1}{2}n_{\vec{x}_i}\left(n_{\vec{x}_i}-1\right)$. This makes the total potential energy in the state under consideration equal to $\frac{1}{2}\sum_{i,j}V(\vec{x}_i,\vec{x}_j)n_{\vec{x}_i}n_{\vec{x}_j}-\frac{1}{2}\sum_i V(\vec{x}_i,\vec{x}_i)n_{\vec{x}_i}$.

Using the same logic as in the case of the ideal gas, it is now easy to write down the operator of the total potential energy in terms of operators $\hat{n}_{\vec{x}}=a^+_{\vec{x}}a_{\vec{x}}$. Using the commutation relations for the creation and annihilation operators the resulting expression can be simplified to the form[25]

$$H_{\text{pair}}=\frac{1}{2}\int d^3x\; d^3y\; V\left(\vec{x},\vec{y}\right)\; a^+_{\vec{x}}a^+_{\vec{y}}a_{\vec{y}}a_{\vec{x}}$$

Note the order of the creation and annihilation operators, which is mandatory. It embodies the above mentioned subtlety.

As we have seen before, the x-representation is usually not the most suitable for the ideal gas Hamiltonian. To have the complete Hamiltonian of a system with the pair interaction presented in a single representation, it is useful to rewrite the potential energy operator H_{pair} in the other representation.

Free particles. All one needs is $a_{\vec{x}}=\int d^3p\,\langle\vec{x}|\vec{p}\rangle\,a_{\vec{p}}=\int d^3p\frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{p}.\vec{x}/\hbar}a_{\vec{p}}$

$$H_{\text{pair}}=\frac{1}{2}\int d^3p_1d^3p_2d^3p_3d^3p_4\;V\left(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4\right)\;a^+_{\vec{p}_1}a^+_{\vec{p}_2}a_{\vec{p}_3}a_{\vec{p}_4}$$

$$V\left(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4\right)=\int\frac{d^3x}{\left(2\pi\hbar\right)^3}\frac{d^3y}{\left(2\pi\hbar\right)^3}V\left(\vec{x},\vec{y}\right)\exp\frac{i\left(\vec{p}_4-\vec{p}_1\right).\vec{x}}{\hbar}\exp\frac{i\left(\vec{p}_3-\vec{p}_2\right).\vec{y}}{\hbar}$$

Periodic external field. The plane waves $\langle\vec{x}|\vec{p}\rangle$ are to be replaced by the Bloch functions $\left\langle\vec{x}|n,\vec{k}\right\rangle=u_n(\vec{k})e^{i\vec{k}.\vec{x}}$.

Spherically symmetric external field. The plane waves $\langle\vec{x}|\vec{p}\rangle$ are to be replaced by the product of a solution of the radial Schrödinger equation and a spherical harmonics $\langle\vec{x}|n,l,m\rangle=R_{nl}\left(x\right)Y_{lm}\left(\varphi,\vartheta\right)$.

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[25] $U=\frac{1}{2}\int d^3x\;d^3y\;V(\vec{x},\vec{y})\;a^+_{\vec{x}}a_{\vec{x}}a^+_{\vec{y}}a_{\vec{y}}-\frac{1}{2}\int d^3x\;V(\vec{x},\vec{x})\;a^+_{\vec{x}}a_{\vec{x}}$
$=\frac{1}{2}\int d^3x\;d^3y\;V(\vec{x},\vec{y})\;a^+_{\vec{x}}\left(\delta\left(x-y\right)\pm a^+_{\vec{y}}a_{\vec{x}}\right)a_{\vec{y}}-\frac{1}{2}\int d^3x\;V(\vec{x},\vec{x})\;a^+_{\vec{x}}a_{\vec{x}}$
$=\pm\frac{1}{2}\int d^3x\;d^3y\;V(\vec{x},\vec{y})\;a^+_{\vec{x}}a^+_{\vec{y}}a_{\vec{x}}a_{\vec{y}}=\frac{1}{2}\int d^3x\;d^3y\;V(\vec{x},\vec{x})\;a^+_{\vec{x}}a^+_{\vec{y}}a_{\vec{y}}a_{\vec{x}}$

Remark: *Even if it is not necessary for our purposes, it is hard to resist the temptation to discuss perhaps the most important pair interaction in the non-relativistic quantum physics, namely the Coulomb interaction. This is of general interest not only because of the vast amount of applications, but also due to the fact that the standard way of dealing with the Coulomb potential in the p-representation involves a mathematical inconsistency. The way in which this inconsistency is treated is in a sense generic, and therefore quite instructive.*
In the x-representation $V_{\text{Coulomb}}(\vec{x},\vec{y}) = \frac{e^2}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}$*, i.e. in the p-representation (which is relevant in the case with no external field)*

$$V_{\text{Coulomb}}(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4) = \frac{e^2}{4\pi}\int \frac{d^3x}{(2\pi)^3}\frac{d^3y}{(2\pi)^3}\frac{1}{|\vec{x}-\vec{y}|}e^{i(\vec{p}_4-\vec{p}_1).\vec{x}}e^{i(\vec{p}_3-\vec{p}_2).\vec{y}}$$

(for the sake of brevity, we use the Heaviside-Lorentz convention in electrodynamics and $\hbar = 1$ *units in QM). This integral, however, is badly divergent. The integrand simply does not drop out fast enough for* $|\vec{x}| \to \infty$ *and* $|\vec{y}| \to \infty$*.*
Instead of giving up the use of the p-representation for the Coulomb potential energy, it is a common habit to use a dirty trick. It starts by considering the Yukawa (or Debey) potential energy $V_{\text{Debey}}(\vec{x},\vec{y}) = \frac{e^2}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}e^{-\mu|\vec{x}-\vec{y}|}$*, for which the p-representation is well defined and can be evaluated readily*[26]

$$V_{\text{Debey}}(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4) = \frac{e^2}{4\pi}\frac{1}{2\pi^2}\frac{1}{\mu^2+(\vec{p}_4-\vec{p}_1)^2}\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$

Now comes the dirty part. It is based on two simple (almost trivial) observations: the first is that $V_{\text{Coulomb}}(\vec{x},\vec{y}) = \lim_{\mu\to 0} V_{\text{Debey}}(\vec{x},\vec{y})$*, and the second is that the limit* $\lim_{\mu\to 0} V_{\text{Debey}}(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)$ *is well defined. From this, a brave heart can easily conclude that* $V_{\text{Coulomb}}(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4) = \frac{e^2}{8\pi^3}\frac{1}{(\vec{p}_4-\vec{p}_1)^2}\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$*. And, believe it or not, this is indeed what is commonly used as the Coulomb potential energy in the p-representation.*
Needless to say, from the mathematical point of view, this is an awful heresy (called illegal change of order of a limit and an integral). How does it come about that physicists are working happily with something so unlawful?
The most popular answer is that the Debey is nothing else but a screened Coulomb, and that in most systems this is more realistic than the pure Coulomb. This is a reasonable answer, with a slight off-taste of a cheat (the limit $\mu \to 0$ *convicts us that we are nevertheless interested in the pure Coulomb).*
Perhaps a bit more fair answer is this one: For μ *small enough, one cannot say experimentally the difference between the Debey and Coulomb. And the common plausible belief is that measurable outputs should not depend on immeasurable inputs (if this was not typically true, the whole science would hardly be possible). If the mathematics nevertheless reveals inconsistencies for some values of an immeasurable parameter, one should feel free to choose another value, which allows for mathematically sound treatment.*

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[26] For $\vec{r} = \vec{x}-\vec{y}$, $V_{\text{Yukawa}}(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4) = \frac{e^2}{4\pi}\int \frac{d^3r}{(2\pi)^3}\frac{d^3y}{(2\pi)^3}\frac{1}{r}e^{-\mu r}e^{i(\vec{p}_4-\vec{p}_1).\vec{r}}e^{i(\vec{p}_3-\vec{p}_2+\vec{p}_4-\vec{p}_1).\vec{y}}$
$= \frac{e^2}{4\pi}\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\int \frac{d^3r}{(2\pi)^3}\frac{e^{-\mu r}}{r}e^{i\vec{q}.\vec{r}}$, where $\vec{q} = \vec{p}_4-\vec{p}_1$. The remaining integral in the spherical coordinates: $\frac{1}{(2\pi)^2}\int_0^\infty dr\, r\, e^{-\mu r}\int_{-1}^1 d\cos\vartheta\, e^{iqr\cos\vartheta} = \frac{1}{2\pi^2}\frac{1}{q}\int_0^\infty dr\, e^{-\mu r}\sin qr$. For the integral $I = \int_0^\infty dr\, e^{-\mu r}\sin qr$ one obtains by double partial integration the relation $I = \frac{q}{\mu^2} - \frac{q^2}{\mu^2}I$, and putting everything together, one comes to the quoted result.

Hamiltonian of a system of unstable particles

Let us consider a system of particles of three types A, B and C, one of which decays into other two, say $C \to AB$. The decay can be understood as the annihilation of the decaying particle and the simultaneous creation of the products. The corresponding interaction Hamiltonian, i.e. the part of the time-evolution generator responsible for this kind of change, is almost self-evident. It should contain the combination $a_i^+ b_j^+ c_k$, where the lowercase letters for creation and annihilation operators correspond to the uppercase letters denoting the type of particle, and subscripts specify the states of the particles involved.

The decay is usually allowed for various combinations of the quantum numbers i, j, k, so the interaction Hamiltonian should assume the form of the sum of all legal alternatives. This is usually written as the sum of all alternatives, each of them multiplied by some factor, which vanishes for the forbidden combinations of quantum numbers: $\sum_{i,j,k} g_{ijk} a_i^+ b_j^+ c_k$.

There is still one problem with this candidate for the interaction Hamiltonian: it is not Hermitian. But this is quite easy to take care of, one just adds the Hermitian conjugate operator $g_{ijk}^* c_k^+ b_j a_i$. So the Hermiticity of the Hamiltonian requires, for any decay, the existence of the reverse process $AB \to C$. All in all

$$H_{\text{decay}} = \sum_{i,j,k} g_{ijk} a_i^+ b_j^+ c_k + g_{ijk}^* c_k^+ b_j a_i$$

Generalizations (decays with 3 or more products, or even some more bizarre processes, with more than one particle annihilated) are straightforward.

Remark: *The factor g_{ijk} is usually called the coupling constant, although it depends on the quantum numbers i, j, k. The reason is that most of decays are local and translation invariant (they do not vary with changes in position). In the x-representation the locality means that $g_{\vec{x},\vec{y},\vec{z}} = g_{\vec{x}} \delta(\vec{x} - \vec{y}) \delta(\vec{x} - \vec{z})$ and translational invariance requires that g does not depend even on $\vec{x}$*

$$H_{\text{decay}} = \int d^3x \; g a_{\vec{x}}^+ b_{\vec{x}}^+ c_{\vec{x}} + g^* c_{\vec{x}}^+ b_{\vec{x}} a_{\vec{x}}$$

Remark: *When dealing with various types of particles, one needs a Fock space which contains the Fock spaces of all particle species under consideration. The obvious first choice is the direct product of the corresponding Fock spaces, e.g. $H = H_A \otimes H_B \otimes H_C$. In such a case any creation/annihilation operator for one particle type commutes with any c/a operator for any other particle type.*
Sometimes, however, it is formally advantageous to modify the commutation relations among different particle species. Although the bosonic c/a operators are always chosen to commute with all the other c/a operators, for fermions it may be preferable to have the c/a operators for different fermions anti-commuting with each other. The point is that sometimes different particle species may be viewed just as different states of the same particle (due to isospin, eightfold way, etc. symmetries). If so, it is clearly favorable to have the (anti)commutation rules which do not need a radical modification with every change of viewpoint.
The question is, of course, if we can choose the (anti)commutation relations at our will. The answer is affirmative. It just requires the appropriate choice of the anti-symmetrized subspace of the direct product of the Fock spaces.

any linear operator expressed in terms of a^+, a

The only purpose of this paragraph is to satisfy a curious reader (if there is any). It will be of no practical importance to us.

First of all, it is very easy to convince oneself that any linear operator can be expressed in terms of creation operators a_i^+, annihilation operators a_i and the vacuum projection operator $|0\rangle\langle 0|$. Indeed, if $\hat{A}$ is a linear operator, then

$$\hat{A} = \sum_{\substack{i,j,\ldots \\ m,n,\ldots}} A_{ij\ldots,kl\ldots} a_i^+ a_j^+ \ldots |0\rangle\langle 0| a_k a_l \ldots$$

where $A_{ij\ldots,kl\ldots} = \frac{\langle i,j,\ldots|\hat{A}|k,l,\ldots\rangle}{\langle i,j,\ldots|i,j,\ldots\rangle\langle k,l,\ldots|k,l,\ldots\rangle}$. Proof: both LHS and RHS have the same matrix elements for all combinations of basis vectors (check this).

The only question therefore is how to get rid of $|0\rangle\langle 0|$. This is done by induction. First, one expresses the $\hat{A}$ operator only within the 0-particle subspace $\mathcal{H}^0$of the Fock space, where it is nothing else but the multiplication by the constant

$$\hat{A}_0 = \tilde{A}_{0,0} \equiv \langle 0|\hat{A}|0\rangle$$

Then, one expresses the $\hat{A}_1 = \hat{A} - \hat{A}_0$ operator within the 0- and 1-particle subspace $\mathcal{H}^0 \oplus \mathcal{H}^1$. Here one gets (check it)

$$\hat{A}_1 = \tilde{A}_{i,j} a_i^+ a_j + \tilde{A}_{i,0} a_i^+ + \tilde{A}_{0,j} a_j$$

where $\tilde{A}_{ij} = \langle i|\hat{A} - \hat{A}_0|j\rangle$, $\tilde{A}_{i,0} = \langle i|\hat{A} - \hat{A}_0|0\rangle$ and $\tilde{A}_{0,j} = \langle 0|\hat{A} - \hat{A}_0|j\rangle$. If one restricts oneself to $\mathcal{H}^0 \oplus \mathcal{H}^1$, then $\hat{A} = \hat{A}_0 + \hat{A}_1$ (why?). So we have succeeded in writing the operator $\hat{A}$ in terms of a_i^+, a_i, even if only in the subspace of the Fock space. This subspace is now expanded to $\mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \mathcal{H}^2$, etc.

It may be instructive now to work out the operator $\hat{A}_2 = \hat{A} - \hat{A}_0 - \hat{A}_1$within $\mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \mathcal{H}^2$ in terms of a_i^+, a_i (try it). We will, however, proceed directly to the general case of $\hat{A}_n = \hat{A} - \sum_{m=0}^{n-1} \hat{A}_m$ within $\bigoplus_{m=0}^{n} \mathcal{H}^m$

$$\hat{A}_n = \sum_{\substack{\text{allowed} \\ \text{combinations}}} \tilde{A}_{ij\ldots,kl\ldots} a_i^+ a_j^+ \ldots a_k a_l \ldots$$

$$\tilde{A}_{ij\ldots,kl\ldots} = \frac{\langle i,j,\ldots|\hat{A} - \sum_{m=0}^{n-1}\hat{A}_m|k,l,\ldots\rangle}{\langle i,j,\ldots|i,j,\ldots\rangle\langle k,l,\ldots|k,l\ldots\rangle}$$

and the "allowed combinations" are either $\underbrace{ij\ldots}_{n}, \underbrace{kl\ldots}_{m\leq n}$ or $\underbrace{ij\ldots}_{m\leq n}, \underbrace{kl\ldots}_{n}$.

If restricted to $\bigoplus_{m=0}^{n} \mathcal{H}^m$, then $\hat{A} = \sum_{m=0}^{n} \hat{A}_m$, i.e. we have $\hat{A}$ expressed in terms of a_i^+, a_i. To get an expression valid not only in subspaces, one takes

$$\hat{A} = \sum_{m=0}^{\infty} \hat{A}_m$$

1.2.3 Calculation of matrix elements — the main trick

One of the most valuable benefits of the use of the creation and annihilation operator formalism is the completely automatous way of matrix elements calculation. The starting point is twofold:
• any ket (bra) vector can be written as a superposition of the basis vectors, which in turn can be obtained by a_i^+ (a_i) operators acting on $|0\rangle$
• any linear operator can be written as a linear combination of products of the a_i^+ and a_i operators
Consequently, any matrix element of a linear operator is equal to some linear combination of the vacuum expectation values (VEVs) of the products of the creation and annihilation operators.

Some of the VEVs are very easy to calculate, namely those in which the last (first) operator in the product is the annihilation (creation) one. Indeed, due to $a_i|0\rangle = 0$ and $\langle 0| a_i^+ = 0$, any such VEV vanishes. Other VEVs are easily brought to the previous form, one just has to use the (anti)commutation relations $\left[a_i, a_j^+\right]_\mp = \delta_{ij}$ to push the creation (annihilation) operators to the right (left). By repeated use of the (anti)commutation relations, the original VEV is brought to the sum of scalar products $\langle 0|0\rangle$ multiplied by pure numbers, and the VEVs vanishing because of $\langle 0| a_i^+ = 0$ or $a_i|0\rangle = 0$. An example is perhaps more instructive than a general exposition.

Example: *Let us consider a decay-like Hamiltonian for just one type of particles* $H_{\mathrm{decay}} = \sum_{i,j,k} g_{ijk}(a_i^+ a_j^+ a_k + a_k^+ a_j a_i)$ *(e.g. phonons, as well as gluons, enjoy this kind of interaction). Note that the coupling "constant" is real* $g_{ijk} = g_{ijk}^*$. *And let us say we want to calculate* $\langle l| H_{\mathrm{decay}} |m,n\rangle$. *First, one writes*

$$\langle l| H_{\mathrm{decay}} |m,n\rangle = \sum_{i,j,k} g_{ijk} \left(\langle 0| a_l a_i^+ a_j^+ a_k a_m^+ a_n^+ |0\rangle + \langle 0| a_l a_k^+ a_j a_i a_m^+ a_n^+ |0\rangle \right)$$

Then one starts to reshuffle the operators. Take, e.g., the first two and use $a_l a_i^+ = \delta_{li} \pm a_i^+ a_l$ *(or with i replaced by k in the second term), to obtain*

$$\begin{aligned}\langle l| H_{\mathrm{decay}} |m,n\rangle = \sum_{i,j,k} g_{ijk} \big(& \delta_{li} \langle 0| a_j^+ a_k a_m^+ a_n^+ |0\rangle \pm \langle 0| a_i^+ a_l a_j^+ a_k a_m^+ a_n^+ |0\rangle \\ & + \delta_{lk} \langle 0| a_j a_i a_m^+ a_n^+ |0\rangle \pm \langle 0| a_k^+ a_l a_j a_i a_m^+ a_n^+ |0\rangle \big)\end{aligned}$$

Three of the four terms have a^+ *next to* $\langle 0|$, *and consequently they vanish. In the remaining term (the third one) one continues with reshuffling*

$$\begin{aligned}\langle l| H_{\mathrm{decay}} |m,n\rangle &= \sum_{i,j,k} g_{ijk}\delta_{lk} \left(\delta_{im} \langle 0| a_j a_n^+ |0\rangle \pm \langle 0| a_j a_m^+ a_i a_n^+ |0\rangle \right) \\ &= \sum_{i,j,k} g_{ijk}\delta_{lk} \left(\delta_{im}\delta_{jn} \pm \delta_{jm} \langle 0| a_i a_n^+ |0\rangle + 0 \right) \\ &= \sum_{i,j,k} g_{ijk}\delta_{lk} \left(\delta_{im}\delta_{jn} \pm \delta_{jm}\delta_{in} \right) = g_{mnl} \pm g_{nml}\end{aligned}$$

The result could be seen directly from the beginning. The point was not to obtain this particular result, but rather to illustrate the general procedure. It should be clear from the example, that however complicated the operator and the states (between which it is sandwiched) are, the calculation proceeds in the same way: one just reshuffles the creation and annihilation operators.

The example has demonstrated an important common feature of this type of calculations: after all the rubbish vanishes, what remains are just products of deltas (Kronecker or Dirac for discrete or continuous cases respectively), each delta originating from some pair of $a\,a^+$. This enables a short-cut in the whole procedure, namely to take into account only those terms in which all a and a^+operators can be organized (without leftovers) in the pairs $a\,a^+$(in this order!), and to assign the appropriate δ to every such pair.

This is usually done within the "clip" notation, like e.g. $\ldots \overbrace{a_i \ldots a_j^+} \ldots$, which tells us that a_i is paired with a_j^+. The factor corresponding to this particular clip is therefore δ_{ij}. In the case of fermions one has to keep track of signs. The reader may try to convince him/herself that the rule is: every pair of clips biting into each other, generates the minus sign for fermions.

Example: *The same example as above, now using the clip short-cut.*

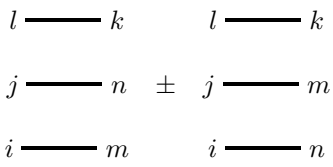
$$\langle l|\, H_{\rm decay}\, |m,n\rangle = \sum_{i,j,k} g_{ijk}\left(\langle 0|\, a_l a_i^+ a_j^+ a_k a_m^+ a_n^+\, |0\rangle + \langle 0|\, a_l a_k^+ a_j a_i a_m^+ a_n^+\, |0\rangle\right)$$

The first term cannot be organized (without leftovers) in pairs of aa^+, so it does not contribute. As to the rest, one has

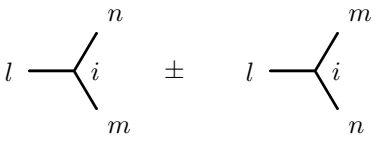
$$\begin{aligned}\langle 0|\, a_l a_k^+ a_j a_i a_m^+ a_n^+\, |0\rangle = \langle 0|\, \overbrace{a_l a_k^+}\, \overbrace{a_j \underbrace{a_i a_m^+}\, a_n^+}\, |0\rangle + \langle 0|\, \overbrace{a_l a_k^+}\, \overbrace{a_j \underbrace{a_i a_m^+}\, a_n^+}\, |0\rangle \\ = \delta_{lk}\delta_{jn}\delta_{im} \pm \delta_{lk}\delta_{jm}\delta_{in}\end{aligned}$$

leading immediately to the same result as before ($g_{mnl} \pm g_{nml}$).

The common habit is to make the short-cut even shorter. Instead of writing the clips above or under the corresponding operators, one draws just the clips and then assigns the corresponding factor to the whole picture. In this comics-like formalism, the matrix element from the example would become



If the interaction is local, the picture is changed slightly. Local interactions contain products of operators in the same position, e.g. $a_{\vec{x}}^+ b_{\vec{x}}^+ c_{\vec{x}}$, which in our discrete case would correspond to $H_{\rm decay} = \sum_i g(a_i^+ a_i^+ a_i + a_i^+ a_i a_i)$. The above picture is then changed to



1.2.4 Feynman diagrams — a comics version of a perturbation theory

The pictures at the end of the previous paragraph may suggest some relation to the Feynman diagrams and indeed they are closely related. Feynman diagrams represent various terms of perturbation series with interaction Hamiltonian written in terms of creation and annihilation operators. We can illustrate this on a simple example of time-independent perturbation theory, where eigenvalues E_n and eigenstates $|\psi_n\rangle$ of a complete Hamiltonian $H = H_0 + \alpha H'$ are expressed in terms of eigenvalues $\mathcal{E}_n$ and eigenstates $|\varphi_n\rangle$ of the unperturbed H_0. This is achieved by expanding the E_n and $|\psi_n\rangle$ in powers of α: $E_n = \mathcal{E}_n + \sum_{k=1}^{\infty} \alpha^k E_n^{(k)}$ and $|\psi_n\rangle = |\varphi_n\rangle + \sum_{k=1}^{\infty} \alpha^k |\psi_n^{(k)}\rangle$, where $H_0 |\varphi_n\rangle = \mathcal{E}_n |\varphi_n\rangle$ and $\langle\varphi_n|\psi_n^{(k)}\rangle = 0$. Comparing coefficients of various powers of α in the equation $(H_0 + \alpha H')|\psi_n\rangle = E_n |\psi_n\rangle$ one obtains explicit formulae for the expansion coefficients. The lowest order results are well known from any textbook on quantum mechanics, in non-degenerate case one gets

$$E_n^{(1)} = \langle\varphi_n| H' |\varphi_n\rangle \qquad E_n^{(2)} = \sum_{m\neq n} \frac{\langle\varphi_n| H' |\varphi_m\rangle \langle\varphi_m| H' |\varphi_n\rangle}{\mathcal{E}_n - \mathcal{E}_m}$$

The higher orders are more involved, e.g.

$$E_n^{(3)} = \sum_{m,m'\neq n} \frac{\langle\varphi_n| H' |\varphi_m\rangle \langle\varphi_m| H' |\varphi_{m'}\rangle \langle\varphi_{m'}| H' |\varphi_n\rangle}{(\mathcal{E}_n - \mathcal{E}_m)(\mathcal{E}_n - \mathcal{E}_{m'})} + \dots$$

where ellipses stand for additional terms, their complexity increases with increasing order.

Interestingly enough, there exists an equivalent, but slightly different version of perturbation theory (see the next remark), in which one obtains the same simple structure in all orders

$$E_n^{(k)} = \sum_{m_1,\dots,m_{k-1}\neq n} \frac{\langle\varphi_n| H' |\varphi_{m_1}\rangle \dots \langle\varphi_{m_{k-1}}| H' |\varphi_n\rangle}{(E_n - \mathcal{E}_{m_1}) \dots (E_n - \mathcal{E}_{m_{k-1}})}$$

Note the presence of E_n (instead of $\mathcal{E}_n$) on the RHS of the last relation. Because of this, the relation is only an implicit one and has to be solved (either iteratively or by expansion of E_n in powers of α) to get the explicit result for $E_n^{(k)}$ (this result is, as it should be, equal to the result of the standard version of the perturbation theory).

The propagators in the Feynman diagrams originate from the same clip trick, although not between the creation and annihilation operators themselves, but rather between some specific combinations thereof. That's why the factor corresponding to the Feynman propagator in the previous section was not just the simple Kronecker or Dirac delta, but rather some more complicated expression.

Finally, we may take yet one step further and indicate where the diagrams come from. In the QM the exponential of an operator plays frequently an important role. The exponential of the interaction Hamiltonian appears in one version of the perturbation theory, the one which became very popular in QFT. The power expansion of the exponential of the local decay Hamiltonian, together with the above example concerning this Hamiltonian, should give the basic taste of the origin of the Feynman diagrams[27].

[27] The reader is encouraged to work out $\langle l| e^{H_{\rm decay}} |m,n\rangle$ up to the second order in g, using all the three (equivalent) methods discussed above (operator reshuffling, clip notation, diagrams). The key part is to calculate $\frac{1}{2}\langle l| \sum_i g(a_i^+ a_i^+ a_i + a_i^+ a_i a_i) \sum_j g(a_j^+ a_j^+ a_j + a_j^+ a_j a_j) |m,n\rangle$.

This rough idea of how the Feynman diagrams arise from the formalism of the creation and annihilation operators in the Fock space is not where we have to stop. We can go much further, at this point we are fully prepared to develop the Feynman diagrams for the non-relativistic many-body QM. Nevertheless, we are not going to.

The reason is that our main goal in these lectures is to develop the relativistic quantum theory. And although the Feynman diagrams in the relativistic and non-relativistic versions have a lot in common, although their derivations are quite similar, and although the applications of the non-relativistic version are important, interesting and instructive, we simply cannot afford to spend too much time on introductions.

1.3 Relativity and Quantum Theory

Actually, as to the quantum fields, the keyword is relativity. Even if QFT is useful also in the nonrelativistic context (see the previous section and the Apeendix), the very notion of the quantum field originated from an endeavor to fit together relativity and quantum theory. This is a nontrivial task: to formulate a relativistic quantum theory is significantly more complicated than it is in a nonrelativistic case. The reason is that specification of a Hamiltonian, the crucial operator of any quantum theory, is much more restricted in relativistic theories.

To understand the source of difficulties, it is sufficient to realize that to have a relativistic quantum theory means to have a quantum theory with measurable predictions which remain unchanged by the relativistic transformations. The relativistic transformations at the classical level are the space-time transformations conserving the interval $ds = \eta_{\mu\nu} dx^\mu dx^\nu$, i.e. boosts, rotations, translations and space-time inversions (the list is exhaustive). They constitute a group.

Now to the quantum level: whenever macroscopic measuring and/or preparing devices enjoy a relativistic transformation, the Hilbert (Fock) space should transform according to a corresponding linear[28] transformation. It is also almost self-evident that the quantum transformation corresponding to a composition of classical transformations should be equal to the composition of quantum transformations corresponding to the individual classical transformations. So at the quantum level the relativistic transformations should constitute a representation of the group of classical transformations.

The point now is that the Hamiltonian, as the time-translations generator, is just one of ten generators of the Poincaré group (boosts, rotations, translations). Consequently, unlike in a nonrelativistic QM, in a relativistic quantum theory one cannot specify the Hamiltonian alone, one has rather to specify it within a complete representation of the Poincaré algebra. This is the starting point of any effort to get a relativistic quantum theory, even if it is not always stated explicitly. The outcome of such efforts are quantum fields. Depending on the philosophy adopted, they use to emerge in at least two different ways. We will call them particle-focused and field-focused.

The particle-focused approach, represented mostly by the Weinberg's book, is very much in the spirit of the previous section. One starts from the Fock space, which is systematically built up from the 1-particle Hilbert space. Creation and annihilation operators are then defined as very natural objects, namely as maps from the n-particle subspace into $(n \pm 1)$-particle ones, and only afterwards quantum fields are built from these operators in a bit sophisticated way (keywords being cluster decomposition principle and relativistic invariance).

The field-focused approach (represented by Peskin–Schroeder, and shared by the majority of textbooks on the subject) starts from the quantization of a classical field, introducing the creation and annihilation operators in this way as quantum incarnations of the normal mode expansion coefficients, and finally providing Fock space as the world for these operators to live in. The logic behind this formal development is nothing else but construction of the Poincaré group generators. So, again, the corner-stone is the relativistic invariance.

[28] Linearity of transformations at the quantum level is necessary to preserve the superposition principle. The symmetry transformations should not change measurable things, which would not be the case if the superposition $|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$ would transform to $T|\psi\rangle \neq c_1 T|\psi_1\rangle + c_2 T|\psi_2\rangle$.

1.3.1 Lorentz and Poincaré groups

This is by no means a systematic exposition to the Lorentz and Poincaré groups and their representations. It is rather a summary of important relations, some of which should be familiar (at some level of rigor) from the previous courses.

the groups

The classical relativistic transformations constitute a group, the corresponding transformations at the quantum level constitute a representation of this group. The (active) group transformations are

$$x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu$$

where Λ^μ_ν are combined space rotations, boosts and space-time inversions, while a^μ describe space-time translations.

The rotations around (and the boosts along) the space axes are

$$R_1(\vartheta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\vartheta & -\sin\vartheta \\ 0 & 0 & \sin\vartheta & \cos\vartheta \end{pmatrix} \qquad B_1(\beta) = \begin{pmatrix} \operatorname{ch}\beta & -\operatorname{sh}\beta & 0 & 0 \\ -\operatorname{sh}\beta & \operatorname{ch}\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_2(\vartheta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & 0 & \sin\vartheta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \qquad B_2(\beta) = \begin{pmatrix} \operatorname{ch}\beta & 0 & -\operatorname{sh}\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\operatorname{sh}\beta & 0 & \operatorname{ch}\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_3(\vartheta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta & 0 \\ 0 & \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B_3(\beta) = \begin{pmatrix} \operatorname{ch}\beta & 0 & 0 & -\operatorname{sh}\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\operatorname{sh}\beta & 0 & 0 & \operatorname{ch}\beta \end{pmatrix}$$

where ϑ is the rotation angle and $\tanh\beta = v/c$. They constitute the Lorentz group. It is a non-compact (because of $\beta \in (-\infty, \infty)$) Lie group.

The translations along the space-time axes are

$$T_0(\alpha) = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad T_1(\alpha) = \begin{pmatrix} 0 \\ \alpha \\ 0 \\ 0 \end{pmatrix} \quad T_2(\alpha) = \begin{pmatrix} 0 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad T_3(\alpha) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{pmatrix}$$

Together with the boosts and rotations they constitute the Poincaré group. It is a non-compact Lie group (on top of the non-compactness of the Lorentz subgroup one has $\alpha \in (-\infty, \infty)$).

The space-time inversions are three different diagonal matrices. The time inversion is given by $T = \operatorname{diag}(-1, 1, 1, 1)$, the space inversion is given by $P = \operatorname{diag}(1, -1, -1, -1)$ and their product is $PT = \operatorname{diag}(-1, -1, -1, -1)$.

the Lie algebra

The standard technique of finding the representations of a Lie group is to find the representations of the corresponding Lie algebra (the commutator algebra of the generators). The standard choice of generators corresponds to the above 10 types of transformations: infinitesimal rotations $R_i(\varepsilon) = 1 - i\varepsilon J_i + \mathcal{O}(\varepsilon^2)$, boosts $B_i(\varepsilon) = 1 - i\varepsilon K_i + \mathcal{O}(\varepsilon^2)$ and translations $T_\mu(\varepsilon) = -i\varepsilon P_\mu + \mathcal{O}(\varepsilon^2)$

$$J_1 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad K_1 = i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J_2 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad K_2 = i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J_3 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$P_0 = i\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad P_1 = i\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad P_2 = i\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad P_3 = i\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Calculation of the commutators is straightforward[29] (even if not very exciting)

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \qquad [J_i, P_0] = 0$$

$$[K_i, K_j] = -i\varepsilon_{ijk}J_k \qquad [J_i, P_j] = i\varepsilon_{ijk}P_k$$

$$[J_i, K_j] = i\varepsilon_{ijk}K_k \qquad [K_i, P_0] = iP_i$$

$$[P_\mu, P_\nu] = 0 \qquad [K_i, P_j] = iP_0\delta_{ij}$$

Remark: *The generators of the Lorentz group are often treated in a more compact way. One writes an infinitesimal transformation as $\Lambda^\mu_{\ \nu} = \eta^\mu_{\ \nu} + \omega^\mu_{\ \nu}$, where $\omega^\mu_{\ \nu}$ is antisymmetric (as can be shown directly form the definition of the Lorentz transformation). Now comes the trick: one writes $\omega^\mu_{\ \nu} = -\frac{i}{2}\omega^{\rho\sigma}(M_{\rho\sigma})^\mu_{\ \nu}$ where $\rho\sigma$ are summation indices, $\mu\nu$ are indices of the Lorentz transformation and $M_{\rho\sigma}$ are chosen so that $\omega^{\rho\sigma}$ are precisely what they look like, i.e. $\eta^{\rho\sigma} + \omega^{\rho\sigma}$ are the Lorentz transformation matrices corresponding to the map of co-vectors to vectors. One can show, after some gymnastics, that*

$$(M_{\rho\sigma})^\mu_{\ \nu} = i\left(\eta^\mu_\rho\eta_{\sigma\nu} - \eta^\mu_\sigma\eta_{\rho\nu}\right) \qquad J_k = \frac{1}{2}\varepsilon_{ijk}M_{ij} \qquad K_i = M_{0i}$$

[29] Commutators between J_i (or K_i) and P_μ follow from $\Lambda^\mu_{\ \nu}(x^\nu + a^\nu) - \left(\Lambda^\mu_{\ \nu}x^\nu + a^\mu\right) = \Lambda^\mu_{\ \nu}a^\nu - a^\mu = (\Lambda - 1)^\mu_{\ \nu}a^\nu$, i.e. one obtains the commutator under consideration directly by acting of the corresponding generator J_i or K_i on the (formal) vector P_μ.

the scalar representation of the Poincaré group

Investigations of the Poincaré group representations are of vital importance to any serious attempt to discuss relativistic quantum theory. It is, however, not our task right now. For quite some time we will need only the simplest representation, the so-called scalar one. All complications introduced by higher representations are postponed to the next parts of the text.

Let us consider a Hilbert space in which some representation of the Poincaré group is defined. Perhaps the most convenient basis of such a space is the one defined by the eigenvectors of the translation generators P_μ, commuting with each other. The eigenvectors are usually denoted as $|p,\sigma\rangle$, where $p=(p_0,p_1,p_2,p_3)$ stands for eigenvalues of P and σ stands for any other quantum numbers. In this notation one has

$$P_\mu |p,\sigma\rangle = p_\mu |p,\sigma\rangle$$

The representation of space-time translations are obtained by exponentiation of generators. If $U(\Lambda,a)$ is the element of the representation, corresponding to the Poincaré transformation $x \to \Lambda x + a$, then $U(1,a) = e^{-iaP}$. And since $|p,\sigma\rangle$ is an eigenstate of P_μ, one obtains

$$U(1,a)\,|p,\sigma\rangle = e^{-ipa}\,|p,\sigma\rangle$$

And how does the representation of the Lorentz subgroup look like? Since the p_μ is a fourvector, one may be tempted to try $U(\Lambda,0)\,|p,\sigma\rangle \stackrel{?}{=} |\Lambda p,\sigma\rangle$. This really works, but only in the simplest, the so-called scalar representation, in which no σ is involved. It is straightforward to check that in such a case the relation

$$U(\Lambda,a)\,|p\rangle = e^{-i(\Lambda p)a}\,|\Lambda p\rangle$$

defines indeed a representation of the Poincaré group.

Let us remark that once the spin is involved, it enters the parameter σ and the transformation is a bit more complicated. It goes like this: $U(\Lambda,a)\,|p,\sigma\rangle = \sum_{\sigma'} e^{-i(\Lambda p)a} C_{\sigma\sigma'}\,|\Lambda p,\sigma'\rangle$ and the coefficients $C_{\sigma\sigma\prime}$ do define the particular representation of the Lorentz group. These complications, however, do not concern us now. For our present purposes the simplest scalar representation will be sufficient.

Now it looks like if we had reached our goal — we have a Hilbert space with a representation of the Poincaré group acting on it. A short inspection reveals, however, that this is just the rather trivial case of the free particle. To see this, it is sufficient to realize that the operator $P^2 = P_\mu P^\mu$ commutes with all the generators of the Poincaré group, i.e. it is a Casimir operator of this group. If we denote the eigenvalue of this operator by the symbol m^2 then the irreducible representations of the Poincaré group can be classified by the value of the m^2. The relation between the energy and the 3-momentum of the state $|p\rangle$ is $E^2 - \vec{p}^2 = m^2$, i.e. for each value of m^2 we really do have the Hilbert space of states of a free relativistic particle. (The reader is encouraged to clarify him/herself how should the Hilbert space of the states of free relativistic particle look like. He/she should come to conclusion, that it has to be equal to what we have encountered just now.)

Nevertheless, this rather trivial representation is a very important one — it becomes the starting point of what we call the particle-focused approach to the quantum field theory. We shall comment on this briefly in the next paragraph.

The Hilbert space spanned over the eigenvectors $|p,\sigma\rangle$ of the translation generators P_μ is not the only possible choice of a playground for the relativistic quantum theory. Another quite natural Hilbert space is provided by the functions $\varphi(x)$. A very simple representation of the Poincaré group is obtained by the mapping

$$\varphi(x) \to \varphi(\Lambda x + a)$$

This means that with any Poincaré transformation $x \to \Lambda x + a$ one simply pulls back the functions in accord with the transformation. It is almost obvious that this, indeed, is a representation (if not, check it in a formal way). Actually, this representation is equivalent to the scalar representation discussed above, as we shall see shortly. (Let us remark that more complicated representations can be defined on n-component functions, where the components are mixed by the transformation in a specific way.)

It is straightforward to work out the generators in this representation (and we shall need them later on). The changes of the space-time position x^μ and the function $\varphi(x)$ by the infinitesimal Poincaré transformation are δx^μ and $\delta x^\mu \partial_\mu \varphi(x)$ respectively, from where one can directly read out the Poincaré generators in this representation

$$\begin{aligned}
\mathcal{J}_i \varphi(x) &= (J_i x)^\mu \, \partial_\mu \varphi(x) \\
\mathcal{K}_i \varphi(x) &= (K_i x)^\mu \, \partial_\mu \varphi(x) \\
\mathcal{P}_\mu \varphi(x) &= i \partial_\mu \varphi(x)
\end{aligned}$$

Using the explicit knowledge of the generators J_i and K_i one obtains

$$\begin{aligned}
\mathcal{J}_i \varphi(x) &= \frac{i}{2} \varepsilon_{ijk} \left(\delta^\mu_j \eta_{k\nu} - \delta^\mu_k \eta_{j\nu} \right) x^\nu \partial_\mu \varphi(x) = -i \varepsilon_{ijk} x_j \partial_k \varphi(x) \\
\mathcal{K}_i \varphi(x) &= i \left(\delta^\mu_i \eta_{0\nu} - \delta^\mu_0 \eta_{i\nu} \right) x^\nu \partial_\mu \varphi(x) = i x_0 \partial_i \varphi(x) - i x_i \partial_0 \varphi(x)
\end{aligned}$$

or even more briefly $\vec{\mathcal{J}} \, \varphi(x) = -i \vec{x} \times \nabla \varphi(x)$ and $\vec{\mathcal{K}} \, \varphi(x) = \; it \nabla \varphi(x) - i \vec{x} \dot{\varphi}(x)$.

At this point the reader may be tempted to interpret $\varphi(x)$ as a wave-function of the ordinary quantum mechanics. There is, however, an important difference between what have now and the standard quantum mechanics. In the usual formulation of the quantum mechanics in terms of wave-functions, the Hamiltonian is specified as a differential operator (with space rather than space-time derivatives) acting on the wave-function $\varphi(x)$. Our representation of the Poincaré algebra did not provide any such Hamiltonian, it just states that the Hamiltonian is the generator of the time translations.

However, if one is really keen to interpret $\varphi(x)$ as the wave-function, one is allowed to do so. Then one may try to specify the Hamiltonian for this irreducible representation by demanding $p^2 = m^2$ for any eigenfunction e^{-ipx}. In this way, one is lead to some specific differential equation for $\varphi(x)$, e.g. to the equation

$$i \partial_t \varphi(x) = \sqrt{m^2 - \partial_i \partial_i} \; \varphi(x)$$

Because of the square root, however, this is not very convenient equation to work with. First of all, it is not straightforward to check, if this operator obeys all the commutation relations of the Poincaré algebra. Second, after the Taylor expansion of the square root one gets infinite number of derivatives, which corresponds to a non-local theory (which is usually quite non-trivial to be put in accord with special relativity). Another ugly feature of the proposed equation is that it treated the time and space derivatives in very different manner, which is at least strange in a would-be relativistic theory. The awkwardness of the square root becomes even more apparent once the interaction with electromagnetic field is considered, but we are not going to penetrate in such details here.

For all these reasons it is a common habit to abandon the above equation and rather to consider the closely related so-called Klein–Gordon equation

$$\left(\partial_\mu \partial^\mu + m^2 \right) \varphi(x) = 0$$

as a kind of a relativistic version of the Schrödinger equation (even if the order of the Schrödinger and Klein–Gordon equations are different).

Note, however, that the Klein–Gordon equation is only related, but not equivalent to the equation with the square root. One of the consequences of this non-equivalence is that the solutions of the Klein-Gordon equation may have both positive and negative energies. This does not pose an immediate problem, since the negative energy solutions can be simply ignored, but it becomes really puzzling, once the electromagnetic interactions are switched on.

Another unpleasant feature is that one cannot interpret $|\varphi(x)|^2$ as a probability density, because this quantity is not conserved. For the Schrödinger equation one was able to derive the continuity equation for the density $|\varphi(x)|^2$ and the corresponding current, but for the Klein–Gordon equation the quantity $|\varphi(x)|^2$ does not obey the continuity equation any more. One can, however, perform with the Klein–Gordon equation a simple massage analogous to the one known from the treatment of the Schrödinger equation, to get another continuity equation with the density $\varphi^*\partial_0\varphi - \varphi\partial_0\varphi^*$. But this density has its own drawback — it can be negative. It cannot play, therefore the role of the probability density.

All this was well known to the pioneers of the quantum theory and eventually led to rejection of wave-function interpretation of $\varphi(x)$ in the Klein–Gordon equation. Strangely enough, the field $\varphi(x)$ equation remained one of the cornerstones of the quantum field theory. The reason is that it was not the function and the equation which were rejected, but rather only their wave-function interpretation.

The function $\varphi(x)$ satisfying the Klein–Gordon is very important — it becomes the starting point of what we call the field-focused approach to the quantum field theory. In this approach the function $\varphi(x)$ is treated as a classical field (transforming according to the considered representation of the Poincaré group) and starting from it one develops step by step the corresponding quantum theory. The whole procedure is discussed in quite some detail in the following chapters.

1.3.2 The logic of the particle-focused approach to QFT

The relativistic quantum theory describes, above all, the physics of elementary particles. Therefore the particle-focused approach looks like the most natural. Nevertheless, it is by far not the most common, for reasons which are mainly historical. Now we have to confess (embarrassed) that in these lectures we are going to follow the less natural, but more wide-spread field-focused approach.[30] The particle-focused approach is only very briefly sketched in this paragraph. *Not everything should and could be understood here, it is sufficient just to catch the flavor.* If too dense and difficult (as it is) the paragraph should be skipped.

One starts with an irreducible representation of the Poincaré group on some 1-particle Hilbert space. The usual basis vectors in the Hilbert space are of the form $|p,\sigma\rangle$, where p is the (overall) momentum of the state and all the other characteristics are included in σ. For a multiparticle state, the σ should contain a continuous spectrum of momenta of particular particles. This provides us with a natural definition of 1-particle states as the ones with discrete σ. In this case it turns out that values of σ correspond to spin (helicity) projections.

Irreducible representations are characterized by eigenvalues of two Casimir operators (operators commuting with all generators), one of them being m^2, the eigenvalue of the Casimir operator P^2, and the second one having to do with the spin. The states in the Hilbert space are therefore characterized by eigenvalues of 3-momentum, i.e. the notation $|\vec{p},\sigma\rangle$ is more appropriate than $|p,\sigma\rangle$ (nevertheless, when dealing with Lorentz transformations, the $|p,\sigma\rangle$ notation is very convenient). The $|\vec{p},\sigma\rangle$ states are still eigenstates of the Hamiltonian, with the eigenvalues $E_{\vec{p}}=\sqrt{\vec{p}^2+m^2}$.

Once a representation of the Poincaré group on a 1-particle Hilbert space is known, one can systematically build up the corresponding Fock space from direct products of the Hilbert ones. The motivation for such a construction is that this would be a natural framework for processes with nonconserved numbers of particles, and such processes are witnessed in the nature. This Fock space benefits from having a natural representation of Poincaré group, namely the one defined by the direct products of the representations of the original 1-particle Hilbert space. The Hamiltonian constructed in this way, as well as all the other generators, correspond to a system of noninteracting particles. In terms of creation and annihilation operators, which are defined as very natural operators in the Fock space the free Hamiltonian has a simple form $H_0=\int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}a^+_{\vec{p}}a_{\vec{p}}$.

The next step, and this is the hard one, is to find another Hamiltonian which would describe, in a relativistic way, a system of interacting particles. One does not start with a specific choice, but rather with a perturbation theory for a generic Hamiltonian $H=H_0+H_{\rm int}$. The perturbation theory is neatly formulated in the interaction picture, where $|\psi_I(t)\rangle = U(t,0)|\psi_I(0)\rangle$, with $U(t,0)$ satisfying $i\partial_t U(t,0)=H_{{\rm int},I}(t)\,U(t,0)$ with the initial condition

[30] The explanation for this is a bit funny.

As to what we call here the particle-centered approach, the textbook is the Weinberg's one. We strongly recommend it to the reader, even if it would mean that he/she will quit these notes. The present author feels that he has nothing to add to the Weinberg's presentation.

But even if the approach of the Weinberg's book is perhaps more natural than any other, it is certainly not a good idea to ignore the traditional development, which we call here the field-centered approach. If for nothing else, then simply because it is traditional and therefore it became a part of the standard background of the majority of particle physicists.

Now as to the textbooks following the traditional approach, quite a few are available. But perhaps in all of them there are points (and unfortunately not just one or two) which are not explained clearly enough, and are therefore not easy to grasp. The aim of the present notes is to provide the standard material with perhaps a bit more emphasis put on some points which are often only glossed over. The hope is, that this would enable reader to penetrate into the subject in combination of a reasonable depth with a relative painlessness.

Nevertheless, beyond any doubt, this hope is not to be fulfilled. The reader will surely find a plenty of disappointing parts in the text.

$U(0,0) = 1$. The perturbative solution of this equation leads to the sum of integrals of the form $\int_{t_0}^{t} dt_1 \dots dt_n \; T\, H_{\mathrm{int},I}(t_1) \dots H_{\mathrm{int},I}(t_n)$, where T orders the factors with respect to decreasing time. For a relativistic theory, these integrals should better be Lorentz invariant, otherwise the scalar products of the time-evolved states would be frame dependent. This presents nontrivial restrictions on the interaction Hamiltonian H_{int}. First of all, the space-time variables should be treated on the same footing, which would suggest an interaction Hamiltonian of the form $H_{\mathrm{int}} = \int d^3x \; \mathcal{H}_{\mathrm{int}}$ and $\mathcal{H}_{\mathrm{int}}$ should be a Lorentz scalar. Furthermore, the T-ordering should not change the value of the product when going from frame to frame, which would suggest $[\mathcal{H}_{\mathrm{int}}(x), \mathcal{H}_{\mathrm{int}}(y)] = 0$ for $(x-y)^2 \leq 0$ (for time-like intervals, the ordering of the Hamiltonians in the T-product is the same in all the reference frames, for space-like intervals the ordering is frame-dependent, but becomes irrelevant for Hamiltonians commuting with each other).

All these requirements do not have a flavor of rigorous statements, they are rather simple observations about how could (would) a relativistic quantum theory look like. It comes as a kind of surprise, that the notion of quantum fields is a straightforward outcome of these considerations. Without going into details, let us sketch the logic of the derivation:

1. As any linear operator, the Hamiltonian can be written as a sum of products of the creation and annihilation operators. The language of the $a^+_{\vec{p}}$ and $a_{\vec{p}}$ operators is technicaly advantageous, e.g. in the formulation of the so-called cluster decomposition principle, stating that experiments which are sufficiently separated in space, have unrelated results.

2. Poincaré transformations of $a^+_{\vec{p}}$ and $a_{\vec{p}}$ (inherited from the transformations of states) are given by $\vec{p}$-dependent matrices, and so the products of such operators (with different momenta) have in general complicated transformation properties. One can, however, combine the $a^+_{\vec{p}}$ and $a_{\vec{p}}$ operators into simply transforming quantities called the creation and annihilation fields $\varphi^+_l(x) = \sum_\sigma \int d^3p \, u_l(x,\vec{p},\sigma) a^+_{\vec{p}}$ and $\varphi^-_l(x) = \sum_\sigma \int d^3p \, v_l(x,\vec{p},\sigma) a_{\vec{p}}$, which are much more suitable for a construction of relativistic quantum theories.

3. The required simple transformation properties of $\varphi^\pm_l$ are the ones independent of any x or $\vec{p}$, namely $\varphi^\pm_l(x) \to \sum_{l'} D_{ll'}(\Lambda^{-1}) \varphi^\pm_{l'}(x)(\Lambda x + a)$, where the D matrices furnish a representation of the Lorentz group. The coefficients u_l and v_l are calculable for any such representation, e.g. for the trivial one $D_{ll'}(\Lambda^{-1}) = 1$ one gets $u(x,\vec{p},\sigma) = \frac{1}{(2\pi)^3\sqrt{2E_{\vec{p}}}} e^{ipx}$ and $v(x,\vec{p},\sigma) = \frac{1}{(2\pi)^3\sqrt{2E_{\vec{p}}}} e^{-ipx}$.

4. One can easily construct a scalar $\mathcal{H}_{\mathrm{int}}(x)$ from the creation and annihilation fields. The vanishing commutator of two such $\mathcal{H}_{\mathrm{int}}(x)$ for time-like intervals, however, is not automatically guaranteed. But if $\mathcal{H}_{\mathrm{int}}(x)$ is constructed from a specific linear combination of the creation and annihilation fields, namely form the fields $\varphi_l(x) = \varphi^+_l(x) + \varphi^-_l(x)$, then the commutator is really zero for time-like intervals. This is the way how the quantum fields are introduced in the Weinberg's approach — as (perhaps the only) the natural objects for construction of interaction Hamiltonians leading to relativistic quantum theories.

1.3.3 The logic of the field-focused approach to QFT

The basic idea of the field-focused approach to quantum fields is to take a classical relativistic field theory and to quantize it canonically (the exact meaning of this statement is to be explained in the next chapter). This makes a perfect sense in case of the electromagnetic field, since the primary task of the canonical quantization is to provide a quantum theory with a given classical limit. If the field is classically well known, but one suspects that there is some underlying quantum theory, then the canonical quantization is a handy tool.

This tool, however, is used also for quantum fields for which there is no such thing as the corresponding classical fields, at least not one observed normally in the nature (the electron-positron field is perhaps the prominent example). This may sound even more surprising after one realizes that there is a well known classical counterpart to the quantum electron, namely the classical electron. So if one is really keen on the canonical quantization, it seems very natural to quantize the (relativistic) classical mechanics of the electron particle, rather than a classical field theory of non-existing classical electron field. But still, what is quantized is indeed the classical field. What is the rationale for this?

First, let us indicate why one avoids the quantization of relativistic particles. Actually even for free particles this would be technically more demanding than it is for free fields. But this is not the main reason in favor of field quantization. The point is that we are not interested in free (particles or field) theory, but rather in a theory with interaction. And while it is straightforward to generalize a relativistic classical free field theory to a relativistic classical field theory with interaction (and then to quantize it), it is quite non-trivial to do so for particles.

Second, it should be perhaps emphasized that the nickname "second quantization", which is sometimes used for the canonical quantization of fields, provides absolutely no clue as to any real reasons for the procedure. On the contrary, the nickname could be very misleading. It suggests that what is quantized is not a classical field, but rather a wave-function, which may be regarded to be the result of (the first) quantization. This point of view just obscures the whole problem and is of no relevance at all (except of, perhaps, the historical one).

So why are the fields quantized? The reason is this: In the non-relativistic quantum theories the dynamics is defined by the Hamiltonian. Important point is that any decent Hamiltonian will do the job. In the relativistic quantum theories, on the other hand, the Hamiltonian, as the time-translations generator, comes in the unity of ten generators of the Poincaré group. Not every decent Hamiltonian defines a relativistic dynamics. The reason is that for a given Hamiltonian, one cannot always supply the nine friends to furnish the Poincaré algebra. As a matter of fact, it is in general quite difficult, if not impossible, to find such nine friends. Usually the most natural way is not to start with the Hamiltonian and then to try to find the corresponding nine generators, but to define the theory from the beginning by presenting the whole set of ten generators[31]. This is definitely much easier to say than to really provide. And here comes the field quantization, a clever trick facilitating simultaneous construction of all ten Poincaré generators.

The starting point is a relativistic classical field theory. This means, first of all, that the Poincaré transformations of the field are well defined (as an example we may take the function $\varphi(x)$ discussed on p. 38, which is now treated as a classical field transforming according to the scalar representation[32] of the Poincaré group). Then only theories which are symmetric under

[31] From this it should be clear that even the formulation, not to speak about solution, of the relativistic quantum theory is about 10 times more difficult than that of non-relativistic quantum theory. The situation is similar to the one in general relativity with 10 components of the metric tensor as opposed to one potential describing the Newtonian gravity.

[32] Why representation, why not any (possibly non-linear) realization? We have answered this question (the

these transformations are considered. Now one could expect that, after the canonical quantization, the Poincaré transformations of classical fields become somehow the desired Poincaré transformations of the Hilbert space of states. The reality, however, is a bit more sophisticated. Here we are going to sketch it only very briefly, details are to be found in the next chapter

At the classical level, to each symmetry there is a conserved charge (Noether's theorem). When formulated in the Hamiltonian formalism, the Poisson brackets of these charges obey the same algebra, as do the Poincaré generators. After canonical quantization, the Noether charges become operators (in the Hilbert space of states), the Poisson brackets become commutators, and the Poisson bracket algebra becomes the Poincaré algebra itself (or, strictly speaking, some representation of the Poincaré algebra). Consequently, the Noether charges become, in the process of the canonical quantization, the generators of the symmetry at the quantum level.

Precisely this is going to be the logic behind the field quantization adopted in these lecture notes: field quantization is a procedure leading in a systematic way to a quantum theory with a consistent package of the ten Poincaré generators.

Let us emphasize once more that another important aspect of canonical quantization, namely that it leads to a quantum theory with a given classical limit, is not utilized here. We ignore this aspect on purpose. In spite of the immense role it has played historically and in spite of the undisputed importance of this aspect in the case of the electromagnetic field, for other fields it is illusory and may lead to undue misconceptions.

To summarize: Enlightened by the third introduction (Relativity and Quantum Theory) we are now going to penetrate a bit into the technique of the canonical quantization of relativistic classical fields. The result will be a relativistic quantum theory in terms of creation and annihilation operators familiar from the second introduction (Many-Body Quantum Mechanics). Clever version of the perturbation theory formulated within the obtained theories will then lead us to the Feynman rules discussed in the first introduction (Conclusions).

Remark: *For the sake of completeness let us mention yet another approach to the quantum field theory — the one which can be naturally called the path integral-focused approach. We will have much to say about it in the chapter* **??**

keyword was the superposition principle) supposing realization in the Hilbert space of quantum states. Now $\varphi(x)$ does not correspond to the quantum state, so it is legitimate to raise the question again.

The answer (pretty unclear at the moment) is that non-linear transformations of classical fields would lead, after quantization, to transformations not conserving the number of particles, which is usually "unphysical" in a sense that one could discriminate between two inertial systems by counting particles.

Chapter 2

Free Scalar Quantum Field

In this chapter the simplest QFT, namely the theory of the free scalar field, is developed along the lines described at the end of the previous chapter. The keyword is the canonical quantization (of the corresponding classical field theory).

2.1 Elements of Classical Field Theory

2.1.1 Lagrangian Field Theory

mass points	nonrelativistic fields	relativistic fields
$q_a(t) \quad a = 1, 2, \ldots$	$\varphi(\vec{x}, t) \qquad \vec{x} \in R^3$	$\varphi(x) \qquad x \in R(3,1)$
$S = \int dt\, L(q, \dot{q})$	$S = \int d^3x\, dt\, \mathcal{L}(\varphi, \nabla\varphi, \dot{\varphi})$	$S = \int d^4x\, \mathcal{L}(\varphi, \partial_\mu\varphi)$
$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0$	$\longrightarrow$	$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} - \frac{\partial \mathcal{L}}{\partial\varphi} = 0$

The third column is just a straightforward generalization of the first one, with the time variable replaced by the corresponding space-time analog. The purpose of the second column is to make such a formal generalization easier to digest[1]. For more than one field $\varphi_a(x)$ $a = 1, \ldots, n$ one has$\mathcal{L}(\varphi_1, \partial_\mu\varphi_1, \ldots, \varphi_n, \partial_\mu\varphi_n)$ and there are n Lagrange-Euler equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_a)} - \frac{\partial \mathcal{L}}{\partial\varphi_a} = 0$$

[1]The dynamical variable is renamed ($q \to \varphi$) and the discrete index is replaced by the continuous one ($q_a \to \varphi_x = \varphi(x)$). The kinetic energy T, in the Lagrangian $L = T - U$, is written as the integral ($\sum_a T(\dot{q}_a) \to \int d^3x\ T(\dot{\varphi}(x))$). The potential energy is in general the double integral ($\sum_{a,b} U(q_a, q_b) \to \int d^3x\, d^3y\, U(\varphi(x), \varphi(y))$), but for the continuous limit of the nearest neighbor interactions (e.g. for an elastic continuum) the potential energy is the function of $\varphi(x)$ and its gradient, with the double integral reduced to the single one ($\int d^3x\, d^3y\, U(\varphi(x), \varphi(y)) \to \int d^3x\, u(\varphi(x), \nabla\varphi(x))$)

The fundamental quantity in the Lagrangian theory is the variation of the Lagrangian density with a variation of fields

$$\begin{aligned}\delta\mathcal{L} &= \mathcal{L}\left(\varphi+\varepsilon\delta\varphi, \partial_\mu\varphi+\varepsilon\partial_\mu\delta\varphi\right) - \mathcal{L}\left(\varphi,\partial_\mu\varphi\right)\\ &= \varepsilon\left[\frac{\partial\mathcal{L}\left(\varphi,\partial_\mu\varphi\right)}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}\left(\varphi,\partial_\mu\varphi\right)}{\partial(\partial_\mu\varphi)}\partial_\mu\delta\varphi\right] + \mathcal{O}(\varepsilon^2)\\ &= \varepsilon\left[\delta\varphi\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right)\right] + \mathcal{O}(\varepsilon^2)\end{aligned}$$

It enters the variation of the action

$$\begin{aligned}\delta S &= \int \delta\mathcal{L}\, d^4x\\ &= \varepsilon\int\left[\delta\varphi\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right)\right] d^4x + \mathcal{O}(\varepsilon^2)\end{aligned}$$

which in turn defines the equations of motion, i.e. the Lagrange–Euler equations for extremal action S (for $\delta\varphi$ vanishing at space infinity always and for initial and final time everywhere)

$$\delta S = 0 \qquad \Rightarrow \qquad \int\delta\varphi\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) d^4x + \int\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\, d^3\Sigma = 0$$

The second term vanishes for $\delta\varphi$ under consideration, the first one vanishes for any allowed $\delta\varphi$ iff $\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0$.

Example: *Free real Klein-Gordon field* $\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$

$$\partial_\mu\partial^\mu\varphi + m^2\varphi = 0$$

This Lagrange-Euler equation is the so-called Klein-Gordon equation. It entered physics as a relativistic generalization of the Schrödinger equation ($\vec{p} = -i\nabla$, $E = i\partial_t$, $\left(p^2 - m^2\right)\varphi = 0$, recall $\hbar = c = 1$). Here, however, it is the equation of motion for some classical field φ.

Example: *Interacting Klein-Gordon field* $\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}g\varphi^4$

$$\partial_\mu\partial^\mu\varphi + m^2\varphi + \frac{1}{3!}g\varphi^3 = 0$$

Nontrivial interactions lead, as a rule, to nonlinear equations of motion.

Example: *Free complex Klein-Gordon field* $\mathcal{L}[\varphi] = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi$
The fields φ^* and φ are treated as independent.[2]

$$\left(\partial_\mu\partial^\mu + m^2\right)\varphi = 0$$
$$\left(\partial_\mu\partial^\mu + m^2\right)\varphi^* = 0$$

The first (second) equation is the Lagrange-Euler equation for $\varphi^*(\varphi)$ respectively.

[2]It seems more natural to write $\varphi = \varphi_1 + i\varphi_2$, where φ_i are real fields and treat these two real fields as independent variables. However, one can equally well take their linear combinations $\varphi_i' = c_{ij}\varphi_j$ as independent variables, and if complex c_{ij} are allowed, then φ^* and φ can be viewed as a specific choice of such linear combinations.

Noether's Theorem

Symmetries imply conservation laws. A symmetry is an (infinitesimal, local) field transformation

$$\varphi(x) \to \varphi(x) + \varepsilon\delta\varphi(x)$$

leaving unchanged either the Lagrangian density $\mathcal{L}$, or (in a weaker version) the Lagrangian $L = \int \mathcal{L}\, d^3x$, or at least (in the weakest version) the action $S = \int \mathcal{L}\, d^4x$. Conservation laws are either local $\partial_\mu j^\mu = 0$ or global $\partial_t Q = 0$, and they hold only for fields satisfying the Lagrange-Euler equations.

symmetry	conservation law	current or charge
$\delta\mathcal{L} = 0$	$\partial_\mu j^\mu = 0$	$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi$
$\delta\mathcal{L} = \varepsilon\partial_\mu\mathcal{J}^\mu(x)$	$\partial_\mu j^\mu = 0$	$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi - \mathcal{J}^\mu$
$\delta L = 0$	$\partial_t Q = 0$	$Q = \int d^3x\, \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\delta\varphi$
$\delta L = \varepsilon\partial_t\mathcal{Q}(t)$	$\partial_t Q = 0$	$Q = \int d^3x\, \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\delta\varphi - \mathcal{Q}$

The first two lines (the strongest version of the Noether's theorem) follow directly from $\delta\mathcal{L}$ given above (supposing the untransformed field φ obeys the Lagrange-Euler equations). The next two lines follow from the $\delta\mathcal{L}$ integrated through the space $\int \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right)\, d^3x = 0$, which can be written as

$$\partial_0 \int d^3x\, \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\delta\varphi = \int d^3x\, \partial_i \left(\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}\delta\varphi\right) = \int dS_i\, \frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}\delta\varphi = 0$$

for the fields obeying the Lagrange-Euler equations and vanishing in the spatial infinity. The conserved quantity has a form of the spatial integral of some "density", but this is not necessary a time-component of a conserved current.

Remark: *For more than one field in the Lagrangian and for the symmetry transformation $\varphi_a(x) \to \varphi_a(x) + \varepsilon\delta_a\varphi(x)$, the conserved current is given by*

$$j^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\delta\varphi_a$$

Proof: $\delta\mathcal{L} = \varepsilon\sum_a \left[\delta\varphi_a\left(\frac{\partial\mathcal{L}}{\partial\varphi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\delta\varphi_a\right)\right] + \mathcal{O}(\varepsilon^2)$.

On the other hand, if more symmetries are involved, i.e. if the Lagrangian density is symmetric under different transformations $\varphi(x) \to \varphi(x) + \varepsilon\delta_k\varphi(x)$, then there is one conserved current for every such transformation

$$j^\mu_k = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta_k\varphi$$

Proof: $\delta\mathcal{L} = \varepsilon\sum \left[\delta\varphi\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta_k\varphi\right)\right] + \mathcal{O}(\varepsilon^2)$.

Example: *Phase change — field transformations* $\varphi \to \varphi e^{-i\alpha}$, $\varphi^* \to \varphi^* e^{i\alpha}$. *Infinitesimal form* $\varphi \to \varphi - i\varepsilon\varphi$, $\varphi^* \to \varphi^* + i\varepsilon\varphi^*$, *i.e.* $\delta\varphi = -i\varphi$ *and* $\delta\varphi^* = i\varphi^*$. *Lagrangian density* $\mathcal{L}[\varphi] = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi - \frac{1}{4}g\left(\varphi^*\varphi\right)^2$. *Symmetry* $\delta\mathcal{L} = 0$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\delta\varphi^* = -i\varphi\partial^\mu\varphi^* + i\varphi^*\partial^\mu\varphi$$
$$Q = \int d^3x\, j^0(x) = i\int d^3x\, \left(\varphi^*\partial^0\varphi - \varphi\partial^0\varphi^*\right)$$

Once the interaction with the electromagnetic field is turned on, this happens to be the electromagnetic current of the Klein-Gordon field.

Example: *Internal symmetries — field transformations* $\varphi_i \to T_{ij}\varphi_j$ $(i,j = 1,\ldots,N)$, *where* $T \in G$ *and* G *is some Lie group of linear transformations. Infinitesimal form* $\varphi_i \to \varphi_i - i\varepsilon_k\left(t_k\right)_{ij}\varphi_j$, *i.e.* $\delta_k\varphi_i = -i\left(t_k\right)_{ij}\varphi_j$. *Lagrangian density* $\mathcal{L}[\varphi_1,\ldots,\varphi_N] = \frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i - \frac{1}{2}m^2\varphi_i^2 - \frac{1}{4}g\left(\varphi_i\varphi_i\right)^2$. *Symmetry* $\delta\mathcal{L} = 0$

$$j_k^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta_k\varphi_i = -i\left(\partial^\mu\varphi_i\right)\left(t_k\right)_{ij}\varphi_j$$
$$Q_k = \int d^3x\, j^0(x) = -i\int d^3x\, \dot{\varphi}_i\left(t_k\right)_{ij}\varphi_j$$

Example: *Space-time translations — field transformations* $\varphi(x) \to \varphi(x+a)$ *(four independent parameters* a_ν *will give four independent conservation laws). Infinitesimal transformations* $\varphi(x) \to \varphi(x) + \varepsilon_\nu\partial^\nu\varphi(x)$, *i.e.* $\delta^\nu\varphi(x) = \partial^\nu\varphi(x)$. *The Lagrangian density* $\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}g\varphi^4$ *as a specific example, but everything holds for any scalar Lagrangian density. Symmetry* $\delta\mathcal{L} = \varepsilon_\nu\partial^\nu\mathcal{L}$ *(note that a scalar Lagrangian density is transformed as* $\mathcal{L}(x) \to \mathcal{L}(x+a)$, *since this holds for any scalar function of* x*). Technically more suitable form of the symmetry* $\delta\mathcal{L} = \varepsilon_\nu\partial_\mu\mathcal{J}^{\mu\nu}(x) = \varepsilon_\nu\partial_\mu\eta^{\mu\nu}\mathcal{L}(x)$.

$$j^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta^\nu\varphi - \mathcal{J}^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L}$$
$$Q^\nu = \int d^3x\, j^{0\nu}(x) = \int d^3x\, \left(\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\partial^\nu\varphi - \eta^{0\nu}\mathcal{L}\right)$$

The conserved quantity is the energy-momentum

$$Q^0 = E = \int d^3x\, \left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\dot{\varphi} - \mathcal{L}\right) \qquad Q^i = P^i = \int d^3x\, \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\,\partial^i\varphi$$

which in our specific example gives

$$Q^0 = E = \frac{1}{2}\int d^3x\, \left(\dot{\varphi}^2 + |\nabla\varphi|^2 + m^2\varphi^2 + \frac{1}{12}g\varphi^4\right)$$
$$\vec{Q} = \vec{P} = \int d^3x\, \dot{\varphi}\,\nabla\varphi$$

Example: *Lorentz transformations — field transformations* $\varphi(x) \to \varphi(\Lambda x)$. *Infinitesimal transformations* $\varphi(x) \to \varphi(x) - \frac{i}{2}\omega^{\rho\sigma}\left(M_{\rho\sigma}\right)^\mu{}_\nu\, x^\nu\partial_\mu\varphi(x)$ *(sorry)*[3], *i.e.* $\delta_{\rho\sigma}\varphi = -i\left(M_{\rho\sigma}\right)^\mu{}_\nu\, x^\nu\partial_\mu\varphi =$

[3]For an explanation of this spooky expression see 1.3.1. Six independent parameters $\omega^{\rho\sigma}$ correspond to 3 rotations (ω^{ij}) and 3 boosts (ω^{0i}). The changes in φ due to these six transformations are denoted as $\delta_{\rho\sigma}\varphi$ with $\rho < \sigma$.

$\left(\delta^\mu_\rho \eta_{\sigma\nu} - \delta^\mu_\sigma \eta_{\rho\nu}\right) x^\nu \partial_\mu \varphi = \left(x_\rho \partial_\sigma - x_\sigma \partial_\rho\right) \varphi$. *The Lagrangian density* $\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}g\varphi^4$ *again only as a specific example, important is that everything holds for any scalar Lagrangian density. Symmetry* $\delta\mathcal{L} = -\frac{i}{2}\omega^{\rho\sigma}\left(M_{\rho\sigma}\right)^\lambda_{\ \nu} x^\nu \partial_\lambda \mathcal{L} = \omega^{\lambda\nu} x_\nu \partial_\lambda \mathcal{L}$, *is processed further to* $\delta\mathcal{L} = \omega^{\lambda\nu}\partial_\lambda\left(x_\nu\mathcal{L}\right) - \omega^{\lambda\nu}\eta_{\lambda\nu}\mathcal{L} = \omega^{\lambda\nu}\partial_\lambda\left(x_\nu\mathcal{L}\right) = \omega^{\lambda\nu}\partial^\mu\left(g_{\lambda\mu}x_\nu\mathcal{L}\right)$. *So one can write* $\delta\mathcal{L} = \omega_{\lambda\nu}\partial_\mu\mathcal{J}^{\mu\lambda\nu}$ *where*[4] $\mathcal{J}^{\mu\lambda\nu} = g^{\lambda\mu}x^\nu\mathcal{L}$

$$j^{\mu\lambda\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\left(x^\nu\partial^\lambda - x^\lambda\partial^\nu\right)\varphi - \eta^{\lambda\mu}x^\nu\mathcal{L}$$

rotations

$$j^{\mu ij} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\left(x^j\partial^i - x^i\partial^j\right)\varphi - \eta^{i\mu}x^j\mathcal{L}$$
$$Q^{ij} = -\int d^3x\,\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\left(x^i\partial^j - x^j\partial^i\right)\varphi$$

boosts

$$j^{\mu 0i} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\left(x^i\partial^0 - x^0\partial^i\right)\varphi - \eta^{0\mu}x^i\mathcal{L}$$
$$Q^{0i} = -\int d^3x\,\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\left(x^0\partial^i - x^i\partial^0\right)\varphi + x^i\mathcal{L}$$

In a slightly different notation

$$\textit{rotations}\qquad \vec{Q}_R = \int d^3x\,\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\vec{x}\times\nabla\varphi$$
$$\textit{boosts}\qquad \vec{Q}_B = t\int d^3x\,\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\nabla\varphi - \int d^3x\;\vec{x}\;\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\dot{\varphi} - \mathcal{L}\right)$$

and finally in our specific example

$$\textit{rotations}\qquad \vec{Q}_R = -\int d^3x\;\dot{\varphi}\;\vec{x}\times\nabla\varphi$$
$$\textit{boosts}\qquad \vec{Q}_B = -t\int d^3x\;\dot{\varphi}\;\nabla\varphi$$
$$+\frac{1}{2}\int d^3x\;\vec{x}\;(\dot{\varphi}^2 + |\nabla\varphi|^2 + m^2\varphi^2 + \frac{1}{12}g\varphi^4)$$

These bunches of letters are not very exciting. The only purpose of showing them is to demonstrate how one can obtain conserved charges for all 10 generators of the Poincarè group. After quantization, these charges will play the role of the generators of the group representation in the Fock space.

[4]The index μ is the standard Lorentz index from the continuity equation, the pair $\lambda\nu$ specifies the transformation, and by coincidence in this case it has the form of Lorentz indices.

2.1.2 Hamiltonian Field Theory

mass points	relativistic fields
$q_a(t)\ ,\ p_a(t) = \frac{\partial L}{\partial \dot{q}_a}$	$\varphi(x)\ ,\ \pi(x) = \frac{\delta L}{\delta \dot{\varphi}(x)} = \frac{\partial \mathcal{L}(x)}{\partial \dot{\varphi}(x)}$
$H = \sum_a \dot{q}_a p_a - L$	$H = \int d^3x\ \left(\dot{\varphi}(x)\,\pi(x) - \mathcal{L}(x)\right)$
$\frac{d}{dt} f = \{H, f\} + \frac{\partial}{\partial t} f$	$\frac{d}{dt} F = \{H, F\} + \frac{\partial}{\partial t} F$
$\{f, g\} = \sum_a \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a} - \frac{\partial g}{\partial p_a}\frac{\partial f}{\partial q_a}$	$\{F, G\} = \int d^3z\ \left(\frac{\delta F}{\delta \pi(z)}\frac{\delta G}{\delta \varphi(z)} - \frac{\delta G}{\delta \pi(z)}\frac{\delta F}{\delta \varphi(z)}\right)$

The only non-trivial issue is the functional derivative $\delta F/\delta\varphi(x)$ which is the generalization of the partial derivative $\partial f/\partial x_n$ (note that in the continuous case φ plays the role of the variable and x plays the role of index, while in the discrete case x is the variable and n is the index). For functions of n variables one has $\delta f[\vec{x}] = f[\vec{x} + \varepsilon\delta\vec{x}] - f[\vec{x}] = \varepsilon\delta\vec{x}.\,\mathrm{grad}\, f + \mathcal{O}(\varepsilon^2) = \sum_n \varepsilon\delta x_n . \partial f/\partial x_n + \mathcal{O}(\varepsilon^2)$. For functionals[5], i.e. for functions with continuous infinite number of variables

$$\delta F[\varphi] = F[\varphi + \varepsilon\delta\varphi] - F[\varphi] = \int dx\ \varepsilon\delta\varphi(x)\ \frac{\delta F[\varphi]}{\delta\varphi(x)} + \mathcal{O}(\varepsilon^2)$$

Clearly $\frac{\delta FG}{\delta\varphi(x)} = \frac{\delta F}{\delta\varphi(x)}G + F\frac{\delta G}{\delta\varphi(x)}$ and $\frac{\delta f(G[\varphi])}{\delta\varphi(x)} = \frac{df[G]}{dG}\frac{\delta G}{\delta\varphi(x)}$, which are the basic properties of anything deserving the name derivative.

For our purposes, the most important functionals are going to be of the form $\int dy\ f(\varphi, \partial_y\varphi)$, where $\partial_y \equiv \frac{\partial}{\partial y}$. In such a case one has[6]

$$\frac{\delta}{\delta\varphi(x)}\int dy\ f(\varphi(y), \partial_y\varphi(y)) = \frac{\partial f(\varphi(x), \partial_y\varphi(x))}{\partial\varphi(x)} - \partial_y \frac{\partial f(\varphi(x), \partial_y\varphi(x))}{\partial(\partial_y\varphi(x))}$$

For 3-dimensional integrals in field Lagrangians this reads

$$\frac{\delta}{\delta\varphi(x)}\int d^3y\ f(\varphi, \dot{\varphi}, \nabla\varphi) = \frac{\partial f(\varphi, \dot{\varphi}, \nabla\varphi)}{\partial\varphi} - \partial_i \frac{\partial f(\varphi, \dot{\varphi}, \nabla\varphi)}{\partial(\partial_i\varphi)}$$
$$\frac{\delta}{\delta\dot{\varphi}(x)}\int d^3y\ f(\varphi, \dot{\varphi}, \nabla\varphi) = \frac{\partial f(\varphi, \dot{\varphi}, \nabla\varphi)}{\partial\dot{\varphi}}$$

where RHS are evaluated at the point x.

As illustrations one can take $\frac{\delta L}{\delta\dot{\varphi}(x)} = \frac{\partial\mathcal{L}(x)}{\partial\dot{\varphi}(x)}$ used in the table above, and $\frac{\delta}{\delta\varphi}\int d^3x\,|\nabla\varphi|^2 = -2\nabla\nabla\varphi = -2\triangle\varphi$ used in the example below.

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[5] Functional is a mapping from the set of functions to the set of numbers (real or complex).

[6] $\delta\int dy\ f(\varphi, \partial_y\varphi) = \int dy\ f(\varphi + \varepsilon\delta\varphi, \partial_y\varphi + \varepsilon\partial_y\delta\varphi) - f(\varphi, \partial_y\varphi)$
$= \varepsilon\int dy \frac{\partial f(\varphi,\partial_y\varphi)}{\partial\varphi}\delta\varphi + \frac{\partial f(\varphi,\partial_y\varphi)}{\partial(\partial_y\varphi)}\partial_y\delta\varphi + \mathcal{O}(\varepsilon^2)$
$= \varepsilon\int dy \left(\frac{\partial f(\varphi,\partial_y\varphi)}{\partial\varphi} - \partial_y\frac{\partial f(\varphi,\partial_y\varphi)}{\partial(\partial_y\varphi)}\right)\delta\varphi + \begin{matrix}\text{vanishing}\\ \text{surface term}\end{matrix} + \mathcal{O}(\varepsilon^2)$

Example: *Klein-Gordon field* $\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 \quad \left(-\frac{1}{4!}g\varphi^4\right)$

$$\pi(x) = \frac{\partial\mathcal{L}(x)}{\partial\dot{\varphi}(x)} = \dot{\varphi}(x) \qquad H = \int d^3x\,\mathcal{H}(x)$$
$$\mathcal{H}(x) = \dot{\varphi}(x)\,\pi(x) - \mathcal{L}(x) = \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2$$
$$\dot{\varphi} = \{H,\varphi\} = \pi \qquad \dot{\pi} = \{H,\pi\} = \triangle\varphi - m^2\varphi$$

Inserting the last relation to the time derivative of the second last one obtains the Klein-Gordon equation $\ddot{\varphi} - \triangle\varphi + m^2\varphi = 0$

Poisson brackets of Noether charges

The conserved Noether charges have an important feature, which turns out to be crucial for our development of QFT: their Poisson brackets obey the Lie algebra of the symmetry group. That is why after the canonical quantization, which transfers functions to operators in a Hilbert space and Poisson brackets to commutators, the Noether charges become operators obeying the Lie algebra of the symmetry group. As such they are quite natural choice for the generators of the group representation in the Hilbert space.

The proof of the above statement is straightforward for internal symmetries. The infinitesimal internal transformations are $\delta_k\varphi_i = -i\,(t_k)_{ij}\,\varphi_j$, where t_k are the generators of the group, satisfying the Lie algebra $[t_i,t_j] = if_{ijk}t_k$. The Poisson brackets of the Noether charges are

$$\begin{aligned}
\{Q_k,Q_l\} &= \left\{\int d^3x\;\pi_i(x)\delta_k\varphi_i(x), \int d^3y\;\pi_m(y)\delta_l\varphi_m(y)\right\} \\
&= -\,(t_k)_{ij}\,(t_l)_{mn}\left\{\int d^3x\;\pi_i(x)\varphi_j(x), \int d^3y\;\pi_m(y)\varphi_n(y)\right\} \\
&= -\int d^3z\sum_a (t_k)_{ij}\,(t_l)_{mn}\,(\delta_{ia}\varphi_j(z)\pi_m(z)\delta_{na} - \delta_{ma}\varphi_n(z)\pi_i(z)\delta_{ja}) \\
&= -\int d^3z\;\pi(z)\,[t_l,t_k]\,\varphi(z) = \int d^3z\;\pi(z) if_{klm}t_m\varphi(z) \\
&= if_{klm}Q_m
\end{aligned}$$

For the Poincarè symmetry the proof is a bit more involved. First of all, the generators now contain derivatives, but this is not a serious problem. For the space translations and rotations, e.g., the proof is just a continuous index variation of the discrete index proof for the internal symmetries, one just writes $P^i = \int d^3x\;d^3y\;\pi(y)\,t^i(y,x)\varphi(x)$ and $Q^{ij} = \int d^3x\;d^3y\;\pi(y)\,t^{ij}(y,x)\varphi(x)$ where $t^i(x,y) = \delta^3(x-y)\partial^i$ and $t^{ij}(x,y) = \delta^3(x-y)(x^j\partial^i - x^i\partial^j)$.

For the time-translation and boosts the generators are not linear in π and φ, the universal proof for such generators is a bit tricky[7]. But for any particular symmetry one may prove the statement just by calculating the Poisson brackets for any pair of generators. In the case of the Poincarè group this "exciting" exercise is left to the reader[8].

[7] P. Ševera, private communication.

[8] Just kidding, the reader has probably better things to do and he or she is invited to take the statement for granted even for the Poincarè group.

2.2 Canonical Quantization

2.2.1 The procedure

The standard way of obtaining a quantum theory with a given classical limit[9]:

$$\begin{array}{c}
\text{classical mechanics in any formalism} \\
\downarrow \\
\text{classical mechanics in the Hamiltonian formalism} \\
\text{(with Poisson brackets)} \\
\Downarrow \\
\text{replacement of canonical variables by linear operators} \\
\text{replacement of Poisson brackets by commutators } \{f,g\} \to \frac{i}{\hbar}\left[\hat{f},\hat{g}\right] \\
\downarrow \\
\text{explicit construction of a Hilbert space } \mathbb{H} \\
\text{explicit construction of the operators} \\
\downarrow \\
\text{quantum mechanics in the Heisenberg picture}
\end{array}$$

Example: *A particle in a potential* $U(x)$

$$\begin{array}{ccc}
 & L = \frac{m\dot{x}^2}{2} - U(x) & \\
 & \downarrow & \\
 & H = \frac{p^2}{2m} + U(x) & \\
\{p,x\} = 1 & & \frac{dF(x,p)}{dt} = \{H,F\} \\
 & \Downarrow & \\
 & \hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{x}) & \\
[\hat{p},\hat{x}] = -i\hbar & & \frac{dF(\hat{x},\hat{p})}{dt} = \frac{i}{\hbar}\left[\hat{H},F\right] \\
 & \downarrow & \\
 & \mathbb{H} = L^2 & \\
\hat{x}\psi(x) = x\psi(x) & & \hat{p}\psi(x) = -i\hbar\partial_x\psi(x) \\
 & \downarrow & \\
\frac{d\hat{p}}{dt} = \frac{i}{\hbar}\left[\hat{H},\hat{p}\right] = -\partial_x U(x) & & \frac{d\hat{x}}{dt} = \frac{i}{\hbar}\left[\hat{H},\hat{x}\right] = \frac{1}{m}\partial_x
\end{array}$$

When written in the Schrödinger picture, the Schrödinger equation is obtained.

The above example is not very impressive, since the final result (in the Schrödinger picture) is the usual starting point of any textbook on quantum mechanics. More instructive examples are provided by a particle in a general electromagnetic field or by the electromagnetic field itself. The latter has played a key role in the development of the quantum field theory, and is going to be discussed thoroughly later on (QFT II, summer term). Here we are going to concentrate on the scalar field quantization. Even this simpler case is sufficient to illustrate some new conceptual problems, not present in the quantization of ordinary mechanical systems with a finite number of degrees of freedom.

[9] As already mentioned in the first chapter, for our purposes, the classical limit is not the issue. Nevertheless, the technique (of the canonical quantization) is going to be very useful.

Example: *The scalar field*

$$L = \int d^3x \, \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{2}m^2\varphi^2$$

$$\downarrow$$

$$H = \int d^3x \, \tfrac{1}{2}\pi^2 + \tfrac{1}{2}\left|\nabla\varphi\right|^2 + \tfrac{1}{2}m^2\varphi^2$$

$$\{\pi(\vec{x},t), \varphi(\vec{y},t)\} = \delta^3(\vec{x}-\vec{y}) \qquad \frac{dF[\varphi,\pi]}{dt} = \{H,F\}$$

$$\Downarrow\Downarrow$$

$$H = \int d^3x \, \tfrac{1}{2}\hat{\pi}^2 + \tfrac{1}{2}\left|\nabla\hat{\varphi}\right|^2 + \tfrac{1}{2}m^2\hat{\varphi}^2$$

$$[\hat{\pi}(\vec{x},t), \hat{\varphi}(\vec{y},t)] = -i\hbar\delta^3(\vec{x}-\vec{y}) \qquad \frac{dF[\hat{\varphi},\hat{\pi}]}{dt} = \frac{i}{\hbar}\left[\hat{H},F\right]$$

$$\downarrow$$

$$\mathbb{H} = ???$$

to be continued

The problem with the example (the reason why it is not finished): $\mathbb{H}$ is in general a non-separable Hilbert space. Indeed: for one degree of freedom (DOF) one gets a separable Hilbert space, for finite number of DOF one would expect still a separable Hilbert space (e.g. the direct product of Hilbert spaces for one DOF), but for infinite number of DOF there is no reason for the Hilbert space to be separable. Even for the simplest case of countable infinite many spins 1/2 the cardinality of the set of orthogonal states is $2^{\aleph_0} = c$. For a field, being a system with continuously many DOF (with infinitely many possible values each) the situation is to be expected at least this bad.

The fact that the resulting Hilbert space comes out non-separable is, on the other hand, a serious problem. The point is that the QM works the way it works due to the beneficial fact that many useful concepts from linear algebra survive a trip to the countable infinite number of dimensions (i.e. the functional analysis resembles in a sense the linear algebra, even if it is much more sophisticated). For continuously many dimensions this is simply not true any more.

Fortunately, there is a way out, at least for the free fields. The point is that the free quantum field can be, as we will see shortly, naturally placed into a separable playground — the Fock space[10]. This is by no means the only possibility, there are other non-equivalent alternatives in separable spaces and yet other alternatives in non-separable ones. However, it would be everything but wise to ignore this nice option. So we will, together with the rest of the world, try to stick to this fortunate encounter and milk it as much as possible.

The Fock space enters the game by changing the perspective a bit and viewing the scalar field as a system of coupled harmonic oscillators. This is done in the next section. The other possibilities and their relation to the Fock space are initially ignored, to be discussed afterwards.

Scalar Field as Harmonic Oscillators

The linear harmonic oscillator is just a special case of the already discussed example, namely a particle in the potential $U(x)$. One can therefore quantize the LHO just as in the above general

[10]For interacting fields the Fock space is not so natural and comfortable choice any more, but one usually tries hard to stay within the Fock space, even if it involves quite some benevolence as to the rigor of mathematics in use. These issues are the subject-matter of the following chapter.

example (with the particular choice $U(x) = m\omega^2 x^2/2$), but this is not the only possibility.

Let us recall that search of the solution of the LHO in the QM (i.e. the eigenvalues and eigenvectors of the Hamiltonian) is simplified considerably by introduction of the operators a and a^+. Analogous quantities can be introduced already at the classical level[11] simply as

$$a = x\sqrt{\frac{m\omega}{2}} + p\frac{i}{\sqrt{2m\omega}}$$
$$a^+ = x\sqrt{\frac{m\omega}{2}} - p\frac{i}{\sqrt{2m\omega}}$$

The point now is that the canonical quantization can be performed in terms of the variables a and a^+

$$L = \tfrac{m\dot{x}^2}{2} - \tfrac{m\omega^2 x^2}{2}$$

$$\downarrow$$

$$H = \tfrac{p^2}{2m} + \tfrac{m\omega^2 x^2}{2} = \omega a^+ a \quad \text{or} \quad H = \tfrac{\omega}{2}\left(a^+ a + a a^+\right)$$

$$\{a, a^+\} = i \qquad \dot{a} = -i\omega a \qquad \dot{a}^+ = i\omega a^+$$

$$\Downarrow\Downarrow$$

$$H = \omega a^+ a \quad \text{or} \quad H = \tfrac{\omega}{2}\left(a^+ a + a a^+\right)$$

$$[a, a^+] = 1 \qquad \dot{a} = -i\omega a \qquad \dot{a}^+ = i\omega a^+$$

$$\downarrow$$

$$\mathbb{H} = \text{space spanned by } |0\rangle, |1\rangle, \ldots$$

$$a|n\rangle = |n-1\rangle \qquad\qquad a^+|n\rangle = |n+1\rangle$$

Note that we have returned back to the convention $\hbar = c = 1$ and refrained from writing the hat above operators. We have considered two (out of many possible) Hamiltonians equivalent at the classical level, but non-equivalent at the quantum level (the standard QM choice being $H = \omega\left(a^+ a + 1/2\right)$). The basis $|n\rangle$ is orthogonal, but not orthonormal.

The relevance of the LHO in the context of the QFT is given by a "miracle" furnished by the 3D Fourier expansion

$$\varphi(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}.\vec{x}} \varphi(\vec{p}, t)$$

which when applied to the Klein-Gordon equation $\partial_\mu \partial^\mu \varphi(\vec{x}, t) + m^2 \varphi(\vec{x}, t) = 0$ leads to

$$\ddot{\varphi}(\vec{p}, t) + (\vec{p}^2 + m^2)\varphi(\vec{p}, t) = 0$$

[11] The complex linear combinations of $x(t)$ and $p(t)$ are not as artificial as they may appear at the first sight. It is quite common to write the solution of the classical equation of motion for LHO in the complex form as $x(t) = \frac{1}{2}(Ae^{-i\omega t} + Be^{i\omega t})$ and $p(t) = -\frac{im\omega}{2}(Ae^{-i\omega t} - Be^{i\omega t})$. Both $x(t)$ and $p(t)$ are in general complex, but if one starts with real quantities, then $B = A^*$, and they remain real forever. The $a(t)$ is just a rescaled $Ae^{-i\omega t}$: $a(t) = \sqrt{m\omega/2}Ae^{-i\omega t}$.

for any $\vec{p}$. Conclusion: the free classical scalar field is equivalent to the (infinite) system of decoupled LHOs[12], where $\varphi(\vec{p},t)$ plays the role of the coordinate (not necessarily real, even if $\varphi(\vec{x},t)$ is real), $\pi=\dot{\varphi}$ the role of the momentum and $\vec{p}$ the role of the index. Note that m has nothing to do with the mass of the oscillators which all have unit mass and

$$\omega_{\vec{p}}^2=\vec{p}^2+m^2$$

Quantization of each mode proceeds in the standard way described above. At the classical level we define

$$a_{\vec{p}}(t)=\varphi(\vec{p},t)\sqrt{\tfrac{\omega_{\vec{p}}}{2}}+\pi(\vec{p},t)\tfrac{i}{\sqrt{2\omega_{\vec{p}}}}$$
$$A_{\vec{p}}^{+}(t)=\varphi(\vec{p},t)\sqrt{\tfrac{\omega_{\vec{p}}}{2}}-\pi(\vec{p},t)\tfrac{i}{\sqrt{2\omega_{\vec{p}}}}$$

We have used the symbol $A_{\vec{p}}^{+}$ instead of the usual $a_{\vec{p}}^{+}$, since we want to reserve the symbol $a_{\vec{p}}^{+}$ for the complex conjugate to $a_{\vec{p}}$. It is essential to realize that for the "complex oscillators" $\varphi(\vec{p},t)$ there is no reason for $A_{\vec{p}}^{+}(t)$ to be equal to the complex conjugate $a_{\vec{p}}^{+}(t)=\varphi^*(\vec{p},t)\sqrt{\omega_{\vec{p}}/2}-i\pi^*(\vec{p},t)/\sqrt{2\omega_{\vec{p}}}$.

For the real classical field, however, the condition $\varphi(\vec{x},t)=\varphi^*(\vec{x},t)$ implies $\varphi(-\vec{p},t)=\varphi^*(\vec{p},t)$ (check it) and the same holds also for the conjugate momentum $\pi(\vec{x},t)$. As a consequence $a_{\vec{p}}^{+}(t)=A_{-\vec{p}}^{+}(t)$ and therefore one obtains

$$\varphi(\vec{x},t)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}(t)\,e^{i\vec{p}.\vec{x}}+a_{\vec{p}}^{+}(t)\,e^{-i\vec{p}.\vec{x}}\right)$$
$$\pi(\vec{x},t)=\int\frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}}(t)\,e^{i\vec{p}.\vec{x}}-a_{\vec{p}}^{+}(t)\,e^{-i\vec{p}.\vec{x}}\right)$$

Now comes the quantization, leading to commutation relations

$$\left[a_{\vec{p}}(t),a_{\vec{p}'}^{+}(t)\right]=(2\pi)^3\delta(\vec{p}-\vec{p}')\qquad[a_{\vec{p}}(t),a_{\vec{p}'}(t)]=\left[a_{\vec{p}}^{+}(t),a_{\vec{p}'}^{+}(t)\right]=0$$

The reader may want to check that these relations are consistent with another set of commutation relations, namely with $[\varphi(\vec{x},t),\pi(\vec{y},t)]=i\delta^3(\vec{x}-\vec{y})$ and $[\varphi(\vec{x},t),\varphi(\vec{y},t)]=[\pi(\vec{x},t),\pi(\vec{y},t)]=0$ (hint: $\int\frac{d^3x}{(2\pi)^3}e^{-i\vec{k}.\vec{x}}=\delta^3(\vec{k})$).

The "miracle" is not over yet. The free field have turned out to be equivalent to the system of independent oscillators, and this system will now turn out to be equivalent to still another system, namely to the system of free non-teracting relativistic particles. Indeed, the free Hamiltonian

[12]What is behind the miracle: The free field is equivalent to the continuous limit of a system of linearly coupled oscillators. Any such system can be "diagonalized", i.e. rewritten as an equivalent system of decoupled oscillators. For a system with translational invariance, the diagonalization is provided by the Fourier transformation. The keyword is diagonalization, rather than Fourier.

written in terms of $a_{\vec{p}}(t)$ and $a^+_{\vec{p}}(t)$ becomes[13]

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2\right)$$
$$= \int \frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}}\left(a^+_{\vec{p}}(t)\,a_{\vec{p}}(t) + \frac{1}{2}\left[a_{\vec{p}}(t), a^+_{\vec{p}}(t)\right]\right)$$

where the last term is an infinite constant (since $[a_{\vec{p}}(t), a^+_{\vec{p}}(t)] = (2\pi)^3\delta^3(\vec{p}-\vec{p})$). This is our first example of the famous (infinite) QFT skeletons in the cupboard. This one is relatively easy to get rid of (to hide it away) simply by subtracting the appropriate constant from the overall energy, which sounds as a legal step.

Another way leading to the same result is to realize that the canonical quantization does not fix the ordering in products of operators. One can obtain different orderings at the quantum level (where the ordering does matter) starting from the different orderings at clasical level (where it does not). One may therefore choose any of the equivalent orderings at the classical level to get the desired ordering at the quantum level. Then one can postulate that the correct ordering is the one leading to the decent Hamiltonian. Anyway, the standard form of the free scalar field Hamiltonian in terms of creation and annihilation operators is

$$H = \int \frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}}\,a^+_{\vec{p}}(t)\,a_{\vec{p}}(t)$$

This looks pretty familiar. Was it not for the explicit time dependence of the creation and annihilation operators, this would be the Hamiltonian of the ideal gas of free relativistic particles (relativistic because of the relativistic energy $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$). The explicit time dependence of the operators, however, is not an issue — the hamiltonian is in fact time-independent, as we shall see shortly (the point is that the time dependence of the creation and annihilation operators turns out to be $a^+_{\vec{p}}(t) = a^+_{\vec{p}}e^{i\omega_{\vec{p}}t}$ and $a_{\vec{p}}(t) = a_{\vec{p}}e^{-i\omega_{\vec{p}}t}$ respectively).

Still, it is not the proper Hamiltonian yet, since it has nothing to act on. But once we hand over an appropriate Hilbert space, it will indeed become the old friend.

For the relativistic quantum theory (and this is what we are after) the Hamiltonian is not the whole story, one rather needs all 10 generators of the Poincarè group. For the space-translations $\vec{P} = \int d^3x\ \pi\ \nabla\varphi$ one obtains[14]

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3}\,\vec{p}\,a^+_{\vec{p}}(t)\,a_{\vec{p}}(t)$$

[13] The result is based on the following algebraic manipulations

$$\int d^3x\ \pi^2 = -\int \frac{d^3x d^3p d^3p'}{(2\pi)^6}\frac{\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}{2}\left(a_{\vec{p}}(t) - a^+_{-\vec{p}}(t)\right)\left(a_{\vec{p}'}(t) - a^+_{-\vec{p}'}(t)\right)e^{i(\vec{p}+\vec{p}').\vec{x}}$$
$$= -\int \frac{d^3p d^3p'}{(2\pi)^3}\frac{\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}{2}\left(a_{\vec{p}}(t) - a^+_{-\vec{p}}(t)\right)\left(a_{\vec{p}'}(t) - a^+_{-\vec{p}'}(t)\right)\delta^3(\vec{p}+\vec{p}')$$
$$= -\int \frac{d^3p}{(2\pi)^3}\frac{\omega_{\vec{p}}}{2}\left(a_{\vec{p}}(t) - a^+_{-\vec{p}}(t)\right)\left(a_{-\vec{p}}(t) - a^+_{\vec{p}}(t)\right)$$

$$\int d^3x\ |\nabla\varphi|^2 + m^2\varphi^2 = \int \frac{d^3x d^3p d^3p'}{(2\pi)^6}\frac{-\vec{p}.\vec{p}' + m^2}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}\left(a_{\vec{p}}(t) + a^+_{-\vec{p}}(t)\right)\left(a_{\vec{p}'}(t) + a^+_{-\vec{p}'}(t)\right)e^{i(\vec{p}+\vec{p}').\vec{x}}$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{\omega_{\vec{p}}}{2}\left(a_{\vec{p}}(t) + a^+_{-\vec{p}}(t)\right)\left(a_{-\vec{p}}(t) + a^+_{\vec{p}}(t)\right)$$

where in the last line we have used $(\vec{p}^2 + m^2)/\omega_{\vec{p}} = \omega_{\vec{p}}$.

Putting everything together, one obtains the result.

[14] $\vec{P} = \int \frac{d^3p}{(2\pi)^3}\frac{\vec{p}}{2}(a_{\vec{p}}(t) - a^+_{-\vec{p}}(t))(a_{-\vec{p}}(t) + a^+_{\vec{p}}(t))$
$= \int \frac{d^3p}{(2\pi)^3}\frac{\vec{p}}{2}(a_{\vec{p}}(t)\,a_{-\vec{p}}(t) - a^+_{-\vec{p}}(t)\,a_{-\vec{p}}(t) + a_{\vec{p}}(t)\,a^+_{\vec{p}}(t) - a^+_{-\vec{p}}(t)\,a^+_{\vec{p}}(t))$

while for the rotations and the boosts, i.e. for $Q^{ij} = \int d^3x \ \pi \left(x^j \partial^i - x^i \partial^j\right) \varphi$ and $Q^{0i} = \int d^3x \ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \left(x^i \partial^0 - x^0 \partial^i\right) \varphi - x^i \mathcal{L}$ the result is[15]

$$Q^{ij} = i \int \frac{d^3p}{(2\pi)^3} \ a^+_{\vec{p}}(t) \left(p^i \partial^j - p^j \partial^i\right) a_{\vec{p}}(t)$$
$$Q^{0i} = i \int \frac{d^3p}{(2\pi)^3} \ \omega_{\vec{p}} a^+_{\vec{p}}(t) \, \partial^i a_{\vec{p}}(t)$$

where $\partial^i = \partial/\partial p_i$ in these formulae.

The reason why these operators are regarded as good candidates for the Poincarè group generators is that at the classical level their Poisson brackets obey the corresponding Lie algebra. After the canonical quantization they are supposed to obey the algebra as well. This, however, needs a check.

The point is that the canonical quantization does not fix the ordering of terms in products. One is free to choose any ordering, each leading to some version of the quantized theory. But in general there is no guarantee that a particular choice of ordering in the 10 generators will preserve the Lie algebra. Therefore one has to check if his or her choice of ordering did not spoil the algebra. The alert reader may like to verify the Lie algebra of the Poincarè group for the generators as given above. The rest of us may just trust the printed text.

Now both the first and the last term vanish since they are integrals of odd functions, so
$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{2} (a^+_{\vec{p}}(t) \, a_{\vec{p}}(t) + a_{\vec{p}}(t) \, a^+_{\vec{p}}(t)) = \int \frac{d^3p}{(2\pi)^3} \vec{p} (a^+_{\vec{p}}(t) \, a_{\vec{p}}(t) + \frac{1}{2} [a_{\vec{p}}(t) \, , a^+_{\vec{p}}(t)])$
and the last term again vanishes as a symmetric integral of an odd function.

[15] $Q_{ij} = \int \frac{d^3x d^3p d^3p'}{2(2\pi)^6} \sqrt{\omega_{\vec{p}}/\omega_{\vec{p}'}} (a_{\vec{p}}(t) - a^+_{-\vec{p}}(t))(a_{\vec{p}'}(t) + a^+_{-\vec{p}'}(t)) x^j p'^i e^{i(\vec{p}+\vec{p}').\vec{x}} - i \leftrightarrow j$
We start with $a_{\vec{p}} a^+_{-\vec{p}'} = [a_{\vec{p}}, a^+_{-\vec{p}'}] + a^+_{-\vec{p}'} a_{\vec{p}} = (2\pi)^3 \delta(\vec{p} + \vec{p}') + a^+_{-\vec{p}'} a_{\vec{p}}$, where $a_{\vec{p}}$ stands for $a_{\vec{p}}(t)$ etc. The term with the δ-function vanishes after trivial integration. Then we write $x^j p'^i e^{i(\vec{p}+\vec{p}').\vec{x}}$ as $-i p'^i \partial'^j e^{i(\vec{p}+\vec{p}').\vec{x}}$, after which the d^3x integration leads to $\partial'^j \delta^3(\vec{p} + \vec{p}')$ and then using $\int dk \, f(k) \, \partial_k \delta(k) = -\partial_k f(k)|_{k=0}$ one gets
$Q_{ij} = -i \int \frac{d^3p}{2(2\pi)^3} \partial'^j \sqrt{\omega_{\vec{p}}/\omega_{\vec{p}'}} (a^+_{-\vec{p}'} a_{\vec{p}} - a^+_{-\vec{p}} a_{\vec{p}'} + a_{\vec{p}} a_{\vec{p}'} - a^+_{-\vec{p}} a^+_{-\vec{p}'}) p'^i|_{\vec{p}'=-\vec{p}} + i \leftrightarrow j$
Now using the Leibniz rule and symetric×antisymmetric cancelations one obtains
$Q_{ij} = -i \int \frac{d^3p}{2(2\pi)^3} p^i ((\partial^j a^+_{\vec{p}}) a_{\vec{p}} - a^+_{-\vec{p}} \partial^j a_{-\vec{p}} + a_{\vec{p}} \partial^j a_{-\vec{p}} - a^+_{-\vec{p}} \partial^j a^+_{\vec{p}}) + i \leftrightarrow j$
At this point one uses per partes integration for the first term, the substitution $\vec{p} \to -\vec{p}$ for the second term, and the commutation relations (and $\vec{p} \to -\vec{p}$) for the last two terms, to get
$Q_{ij} = i \int \frac{d^3p}{(2\pi)^3} a^+_{\vec{p}} (p^i \partial^j - p^j \partial^i) a_{\vec{p}} - i \int \frac{d^3p}{4(2\pi)^3} (p^i \partial^j - p^j \partial^i)(2 a^+_{\vec{p}} a_{\vec{p}} + a_{\vec{p}} a_{-\vec{p}} - a^+_{\vec{p}} a^+_{-\vec{p}})$
In the second term the dp^i or dp^j integral is trivial, leading to momenta with some components infinite. This term would be therefore important only if states containing particles with infinite momenta are allowed in the game. Such states, however, are considered unphysical (having e.g. infinite energy in the free field case). Therefore the last term can be (has to be) ignored. (One can even show, that this term is equal to the surface term in the x-space, which was set to zero in the proof of the Noether's theorem, so one should set this term to zero as well.)

Boost generators are left as an exercise for the reader.

Side remark on complex fields

For the introductory exposition of the basic ideas and techniques of QFT, the real scalar field is an appropriate and sufficient tool. At this point, however, it seems natural to say a few words also about complex scalar fields. If nothing else, the similarities and differences between the real and the complex scalar fields are quite illustrative. The content of this paragraph is not needed for the understanding of what follows, it is presented here rather for sake of future references.

The Lagrangian density for the free complex scalar fields reads

$$\mathcal{L}\left[\varphi^*,\varphi\right]=\partial_\mu\varphi^*\partial^\mu\varphi-m^2\varphi^*\varphi$$

where $\varphi=\varphi_1+i\varphi_2$. The complex field φ is a (complex) linear combination of two real fields φ_1 and φ_2. One can treat either φ_1 and φ_2, or φ and φ^* as independent variables, the particular choice is just the mater of taste. Usually the pair φ and φ^* is much more convenient.

It is straightforward to check that the Lagrange-Euler equation for φ and φ^* (as well as for φ_1 and φ_2) is the Klein-Gordon equation. Performing now the 3D Fourier transformation of both $\varphi\left(\vec{x},t\right)$ and $\varphi^*\left(\vec{x},t\right)$, one immediately realizes (just like in the case of the real scalar field) that $\varphi\left(\vec{p},t\right)$ and $\varphi^*\left(\vec{p},t\right)$ play the role of the coordinate of a harmonic oscillator

$$\ddot{\varphi}\left(\vec{p},t\right)+(\vec{p}^2+m^2)\varphi\left(\vec{p},t\right)=0$$
$$\ddot{\varphi}^*\left(\vec{p},t\right)+(\vec{p}^2+m^2)\varphi^*\left(\vec{p},t\right)=0$$

while $\pi\left(\vec{p},t\right)=\dot{\varphi}^*\left(\vec{p},t\right)$ and $\pi^*\left(\vec{p},t\right)=\dot{\varphi}\left(\vec{p},t\right)$ play the role of the corresponding momenta. The variable $\vec{p}$ plays the role of the index and the frequency of the oscillator with the index $\vec{p}$ is $\omega_{\vec{p}}^2=\vec{p}^2+m^2$

Quantization of each mode proceeds again just like in the case of the real field. For the φ field one obtains[16]

$$a_{\vec{p}}\left(t\right)=\varphi\left(\vec{p},t\right)\sqrt{\tfrac{\omega_{\vec{p}}}{2}}+\pi^*\left(\vec{p},t\right)\tfrac{i}{\sqrt{2\omega_{\vec{p}}}}$$
$$A_{\vec{p}}^+\left(t\right)=\varphi\left(\vec{p},t\right)\sqrt{\tfrac{\omega_{\vec{p}}}{2}}-\pi^*\left(\vec{p},t\right)\tfrac{i}{\sqrt{2\omega_{\vec{p}}}}$$

but now, on the contrary to the real field case, there is no relation between $A_{\vec{p}}^+$ and $a_{\vec{p}}^+$. It is a common habit to replace the symbol $A_{\vec{p}}^+$ by the symbol $b_{-\vec{p}}^+=A_{\vec{p}}^+$ and to write the fields as[17]

$$\varphi\left(\vec{x},t\right)=\int\frac{d^3p}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}\left(t\right)e^{i\vec{p}.\vec{x}}+b_{\vec{p}}^+\left(t\right)e^{-i\vec{p}.\vec{x}}\right)$$
$$\varphi^*\left(\vec{x},t\right)=\int\frac{d^3p}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}^+\left(t\right)e^{-i\vec{p}.\vec{x}}+b_{\vec{p}}\left(t\right)e^{i\vec{p}.\vec{x}}\right)$$

Now comes the quantization, leading to the commutation relations

$$\left[a_{\vec{p}}\left(t\right),a_{\vec{p}'}^+\left(t\right)\right]=\left(2\pi\right)^3\delta\left(\vec{p}-\vec{p}'\right)$$
$$\left[b_{\vec{p}}\left(t\right),b_{\vec{p}'}^+\left(t\right)\right]=\left(2\pi\right)^3\delta\left(\vec{p}-\vec{p}'\right)$$

[16] For the φ^* field one has the complex conjugated relations
$a_{\vec{p}}^+\left(t\right)=\varphi^*\left(\vec{p},t\right)\sqrt{\omega_{\vec{p}}/2}+i\pi\left(\vec{p},t\right)/\sqrt{2\omega_{\vec{p}}}$ and $A_{\vec{p}}\left(t\right)=\varphi^*\left(\vec{p},t\right)\sqrt{\omega_{\vec{p}}/2}-i\pi\left(\vec{p},t\right)/\sqrt{2\omega_{\vec{p}}}$

[17] For the momenta one has
$\pi\left(\vec{x},t\right)=\int\frac{d^3p}{\left(2\pi\right)^3}\left(-i\right)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}}\left(t\right)e^{i\vec{p}.\vec{x}}-b_{\vec{p}}^+\left(t\right)e^{-i\vec{p}.\vec{x}}\right)$
$\pi^*\left(\vec{x},t\right)=\int\frac{d^3p}{\left(2\pi\right)^3}i\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}}^+\left(t\right)e^{-i\vec{p}.\vec{x}}-b_{\vec{p}}\left(t\right)e^{i\vec{p}.\vec{x}}\right)$

while all the other commutators vanish.

The standard form of the Hamiltonian becomes

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}} \left(a^+_{\vec{p}}(t)\, a_{\vec{p}}(t) + b^+_{\vec{p}}(t)\, b_{\vec{p}}(t)\right)$$

which looks very much like the Hamiltonian of the ideal gas of two types (a and b) of free relativistic particles. The other generators can be obtained just like in the case of the real field.

The main difference with respect to the real field is that now there are two types of particles in the game, with the creation operators $a^+_{\vec{p}}$ and $b^+_{\vec{p}}$ respectively. Both types have the same mass and, as a rule, they correspond to a particle and its antiparticle. This becomes even more natural when the interaction with the electromagnetic field is introduced in the standard way (which we are not going to discuss now). It turns out that the particles created by $a^+_{\vec{p}}$ and $b^+_{\vec{p}}$ have strictly opposite electric charge.

Remark: *At the end of the Introduction No. 2 (many-body quantum mechanics) we have seen that lines in Feynman diagrams originate form reshuffling of creation and annihilation operators in computation of vacuum expectation values of specific operators. Once there are two sets of creation and annihilation operators in the game (like the $a^+_{\vec{p}}$, $a_{\vec{p}}$ and $b^+_{\vec{p}}$, $b_{\vec{p}}$ for the complex scalar field) one should perhaps use two different types of lines (plain, dashed, wiggly, etc.) in diagrams. It is a common habit to use the same type of line for a- and b-operators and to distinguish between them using an arrow. A line with one orientation of the arrow coresponds to the a-operators, while the opposite orientation corresponds to the b-operators.*
So far we have mentioned only one type of lines with arrows in Feynman diagrams, namely the lines corresponding to electrons and positrons. These arrows are present due to the fact that the corresponding field is complex (the presence of two types of creation and annihilation operators $a^+_{\vec{p}}$, $a_{\vec{p}}$ and $b^+_{\vec{p}}$, $b_{\vec{p}}$ is a typical feature of any complex field).
Strangely enough (at least at the first sight), for the scalar complex field the arrows are not used. The reason is that the factor corresponding to the line coming from reshuffling of $a^+_{\vec{p}}$ and $a_{\vec{p}}$ is exactly the same as the factor for the line corresponding to reshuffling of $b^+_{\vec{p}}$ and $b_{\vec{p}}$. So for the scalar field there is no need to distinguish these two cases by arrows or by any other means.

Time dependence of free fields

Even if motivated by the free field case, the operators $a_{\vec{p}}(t)$, $a^+_{\vec{p}}(t)$ can be introduced equally well in the case of interacting fields. The above (so-called equal-time) commutation relations would remain unchanged. Nevertheless, in the case of interacting fields, one is faced with very serious problems which are, fortunately, not present in the free field case.

The crucial difference between the free and interacting field lies in the fact that for the free fields the time dependence of these operators is explicitly known. At the classical level, the independent oscillators enjoy the simple harmonic motion, with the time dependence $e^{\pm i\omega_{\vec{p}}t}$. At the quantum level the same is true, as one can see immediately by solving the equation of motion

$$\dot{a}^+_{\vec{p}}(t) = i\left[H, a^+_{\vec{p}}(t)\right] = i\left[\int \frac{d^3p'}{(2\pi)^3}\omega_{\vec{p}'}\; a^+_{\vec{p}'}(t)\, a_{\vec{p}'}(t), a^+_{\vec{p}}(t)\right] = i\omega_{\vec{p}}\, a^+_{\vec{p}}(t)$$

and $\dot{a}_{\vec{p}}(t) = -i\omega_{\vec{p}}\, a_{\vec{p}}(t)$ along the same lines. From now on, we will therefore write for the free fields

$$a^+_{\vec{p}}(t) = a^+_{\vec{p}} e^{i\omega_{\vec{p}}t}$$
$$a_{\vec{p}}(t) = a_{\vec{p}} e^{-i\omega_{\vec{p}}t}$$

where $a^+_{\vec{p}}$ and $a_{\vec{p}}$ are time-independent creation and annihilation operators (they coincide with $a^+_{\vec{p}}(0)$ and $a_{\vec{p}}(0)$). This enables us to write the free quantum field in a bit nicer way as

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}e^{-ipx} + a^+_{\vec{p}}e^{ipx}\right)$$

where $p^0 = \omega_{\vec{p}}$. For interacting fields there is no such simple expression and this very fact makes the quantum theory of interacting fields such a complicated affair.

Remark: *The problem with the interacting fields is not only merely that we do not know their time dependence explicitly. The problem is much deeper and concerns the Hilbert space of the QFT. In the next section we are going to build the separable Hilbert space for the free fields, the construction is based on the commutation relations for the operators $a_{\vec{p}}(t)$ and $a^+_{\vec{p}}(t)$. Since these commutation relations hold also for the interacting fields, one may consider this construction as being valid for both cases. This, however, is not true.*
The problem is that once the Hilbert space is specified, one has to check whether the Hamiltonian is a well defined operator in this space, i.e. if it defines a decent time evolution. The explicitly known time-dependence of the free fields answers this question for free fields. For the interacting fields the situation is much worse. Not only we do not have a proof for a decent time evolution, on contrary, in some cases we have a proof that the time evolution takes any initial state away from this Hilbert space. We will come back to these issues in the next chapter. Until then, let us enjoy the friendly (even if not very exciting) world of the free fields.

Hilbert Space

The construction of the Hilbert space for any of the infinitely many LHOs representing the free scalar field is straightforward, as described above. Merging all these Hilbert spaces together is also straightforward, provided there is a *common ground state* $|0\rangle$. Once such a state is postulated[18], the overall Hilbert space is built as an infinite direct sum of Hilbert spaces of individual LHOs.

Such a space is simply the space spanned by the basis

$$\begin{aligned} |0\rangle & \\ |\vec{p}\rangle &= \sqrt{2\omega_{\vec{p}}}a^+_{\vec{p}}\,|0\rangle \\ |\vec{p},\vec{p}'\rangle &= \sqrt{2\omega_{\vec{p}'}}a^+_{\vec{p}'}\,|\vec{p}\rangle \\ |\vec{p},\vec{p}',\vec{p}''\rangle &= \sqrt{2\omega_{\vec{p}}}a^+_{\vec{p}}\,|\vec{p}',\vec{p}''\rangle \\ &\vdots \end{aligned}$$

where all creation operators are taken at a fixed time, say $t = 0$, i.e. $a^+_{\vec{p}} \equiv a^+_{\vec{p}}(0)$. The normalization (with notation $E_{\vec{p}} = \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$)

$$\langle\vec{p}|\vec{p}'\rangle = 2E_{\vec{p}}\,(2\pi)^3\,\delta^3\,(\vec{p}-\vec{p}')$$

is Lorentz invariant (without $\sqrt{2E_{\vec{p}}}$ in the definition of $|\vec{p}\rangle$ it would not be). Reason: The integral $\int d^3p\;\delta^3(\vec{p}) = 1$ is Lorentz invariant, while d^3p and $\delta^3(\vec{p})$ individually are not. The ratio E/d^3p, on the other hand, is invariant[19] and so is the $d^3p\;\delta^3(\vec{p})\;E/d^3p = E\;\delta^3(\vec{p})$.

The Hilbert space constructed in this way is nothing else but the Fock space introduced in the first chapter. This leads to another shift in perspective: first we have viewed a classical field as a system of classical oscillators, now it turns out that the corresponding system of quantum oscillators can be viewed as a system of particles. Perhaps surprising, and very important.

Let us remark that the Fock space is a separable Hilbert space. Originally our system looked like having continuously infinite dimensional space of states, nevertheless now the number of dimensions seems to be countable. How come? This question is definitely worth discussing, but let us postpone it until the last section of this chapter.

At this point we can continue with the scalar field quantization. It was interrupted at the point $\mathbb{H}$ =???, where one now takes $\mathbb{H}$ = the Fock space. Once the Hilbert space is given explicitly, the last step is the explicit construction of the relevant operators. As to the Hamiltonian, we know it in terms of creation and annihilation operators already, and so it happened that it is just the Hamiltonian of a system of free noninteracting relativistic particles.

An important consequence of the explicit form of the Poincarè generators are the transformation properties of the basis vectors $|\vec{p}\rangle$. For this purpose the suitable notation is the 4-vector one: instead of $|\vec{p}\rangle$ one writes $|p\rangle$ where $p^0 = \omega_{\vec{p}}$ (dependent variable). The Lorentz transformation takes a simple form

$$|p\rangle \overset{\Lambda}{\to} |\Lambda p\rangle$$

[18]For infinite number of oscillators (unlike for the finite one) the existence of such a state is not guaranteed. One is free to assume its existence, but this is an independent assumption, not following from the commutation relations. We will search more into this issue in a while.

[19]Indeed, let us consider the boost along the x_3 axis, with the velocity β. The Lorentz transformation of a 4-momentum $p = (E,\vec{p})$ is $E \to \gamma E + \gamma\beta p_3$, $p_1 \to p_1$, $p_2 \to p_2$ and $p_3 \to \gamma p_3 + \gamma\beta E$ (where $\gamma = \sqrt{1-\beta^2}$), and the same transformation holds for an infinitesimal 4-vector dp. Clearly, d^3p is not invariant $d^3p \to dp_1dp_2\,(\gamma dp_3 + \gamma\beta dE) = d^3p\,(\gamma + \gamma\beta dE/dp_3)$. For $dE = 0$ this would be just a Lorentz contraction, but if both p and $p+dp$ correspond to the same mass m, then $E = \sqrt{m^2+\vec{p}^2}$ and $dE/dp_3 = p_3/\sqrt{m^2+\vec{p}^2} = p_3/E$. Therefore $d^3p \to d^3p\,(\gamma + \gamma\beta p_3/E)$ and finally $\frac{d^3p}{E} \to \frac{d^3p(\gamma+\gamma\beta p_3/E)}{\gamma E+\gamma\beta p_3} = \frac{d^3p}{E}$

This may seem almost self-evident, but it is not. As we will see in a moment, for another basis (x-representation) where this transformation rule looks equally self-evident, it simply does not hold. The proof of the transformation rule, i.e. the calculation of how the generators act on the states $|p\rangle$, is therefore mandatory. For rotations at, say, $t = 0$ one has

$$\begin{aligned} -iQ^{ij}\,|p\rangle &= \int \frac{d^3p'}{(2\pi)^3} a^+_{\vec{p}'} \left(p'^i\partial'^j - p'^j\partial'^i\right) a_{\vec{p}'}\sqrt{2\omega_{\vec{p}}}a^+_{\vec{p}}\,|0\rangle \\ &= \sqrt{2\omega_{\vec{p}}}\int d^3p'\, a^+_{\vec{p}'}\left(p'^i\partial'^j - p'^j\partial'^i\right)\delta\left(\vec{p}' - \vec{p}\right)|0\rangle \\ &= -\sqrt{2\omega_{\vec{p}}}\left(p^i\partial^j - p^j\partial^i\right)a^+_{\vec{p}}\,|0\rangle = -\left(p^i\partial^j - p^j\partial^i\right)|p\rangle \end{aligned}$$

where in the last step we have used $(p^i\partial^j - p^j\partial^i)\sqrt{2\omega_{\vec{p}}} = 0$. Now the derivative of $|p\rangle$ in a direction k is defined by $|p + \epsilon k\rangle = |p\rangle + \epsilon k_\mu\partial^\mu\,|p\rangle$. For rotations $k = -iJ^k p$ ($k_i = -i(J^k)_{ij}p_j = -\varepsilon_{ijk}p_j$) $\Rightarrow \left|p - i\epsilon J^k p\right\rangle = |p\rangle - \epsilon.\varepsilon_{ijk}p_j\partial_i\,|p\rangle$, i.e.

$$\left(1 - i\epsilon Q^{ij}\right)|p\rangle = \left|\left(1 - i\epsilon J^k\right)p\right\rangle$$

which is an infinitesimal form of the transformation rule for rotations.

For boosts one gets along the same lines

$$\begin{aligned} -iQ^{0i}\,|p\rangle &= \int \frac{d^3p'}{(2\pi)^3}\omega_{\vec{p}'}a^+_{\vec{p}'}\partial'^i a_{\vec{p}'}\sqrt{2\omega_{\vec{p}}}a^+_{\vec{p}}\,|0\rangle \\ &= \sqrt{2\omega_{\vec{p}}}\int d^3p'\,\omega_{\vec{p}'}a^+_{\vec{p}'}\partial'^i\delta\left(\vec{p}' - \vec{p}\right)|0\rangle \\ &= -\sqrt{2\omega_{\vec{p}}}\partial^i\omega_{\vec{p}}a^+_{\vec{p}}\,|0\rangle = -\frac{p^i}{2\omega_{\vec{p}}}\,|\vec{p}\rangle - \omega_{\vec{p}}\partial^i\,|p\rangle \end{aligned}$$

and since $\left|p - i\varepsilon K^i p\right\rangle = |p\rangle - i\epsilon\left(K^i\right)_{jk}p_k\partial_j\,|p\rangle = |p\rangle - \epsilon p_i\partial_0\,|p\rangle - \epsilon p_0\partial_i\,|p\rangle$, one finally obtains (realizing that $\partial_0\,|p\rangle = \partial_0\sqrt{2p_0}a^+_{\vec{p}}\,|0\rangle = \frac{1}{\sqrt{2p_0}}a^+_{\vec{p}}\,|0\rangle = \frac{1}{2p_0}\,|p\rangle$)

$$\left(1 - i\epsilon Q^{0i}\right)|p\rangle = \left|\left(1 - i\epsilon K^i\right)p\right\rangle$$

which is an infinitesimal form of the transformation rule for boosts.

As to the translations, the transformation rule is even simpler

$$|p\rangle \xrightarrow{a} e^{-ipa}\,|p\rangle$$

as follows directly from the explicit form of the translation generators, which implies $P\,|p\rangle = p\,|p\rangle$ (where $P^0 = H$).

So far everything applied only to $t = 0$. However, once the explicit time dependence of the creation and annihilation operators in the free field case is found in the next section, the proof is trivially generalized for any t.

Quasilocalized states

So far, the quantum field have played a role of merely an auxiliary quantity, appearing in the process of the canonical quantization. The creation and annihilation operators look more "physical", since they create or annihilate physically well defined states (of course, only to the extent to which we consider the states with the sharp momentum being well defined). Nevertheless the

fields will appear again and again, so after a while one becomes so accustomed to them, that one tends to consider them to be quite natural objects. Here we want to stress that besides this psychological reason there is also a good physical reason why the quantum fields really deserve to be considered "physical".

Let us consider the Fourier transform of the creation operator

$$a^{+}(x)=\int\frac{d^{3}p}{(2\pi)^{3}}e^{-i\vec{p}.\vec{x}}a_{\vec{p}}^{+}(t)=\int\frac{d^{3}p}{(2\pi)^{3}}e^{ipx}a_{\vec{p}}^{+}$$

Acting on the vacuum state one obtains $a^{+}(x)|0\rangle=\int\frac{d^{3}p}{(2\pi)^{3}}e^{ipx}\frac{1}{\sqrt{2\omega_{-\vec{p}}}}|\vec{p}\rangle$, which is the superposition (of normalized momentum eigenstates) corresponding to the state of the particle localized at the point x. This would be a nice object to deal with, was there not for the unpleasant fact that it is not covariant. The state localized at the point x is in general not Lorentz transformed[20] to the state localized at the point Λx. Indeed

$$a^{+}(x)|0\rangle\rightarrow\int\frac{d^{3}p}{(2\pi)^{3}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}e^{ipx}|\Lambda p\rangle$$

and this is in general not equal to $a^{+}(\Lambda x)|0\rangle$. The problem is the non-invariance of $d^{3}p/\sqrt{2\omega_{\vec{p}}}$. Were it invariant, the substitution $p\rightarrow\Lambda^{-1}p$ would do the job.

Now let us consider $\varphi(x)|0\rangle=\int\frac{d^{3}p}{(2\pi)^{3}}\frac{1}{2E_{p}}e^{ipx}|p\rangle$ where $E_{p}=\omega_{\vec{p}}=p^{0}$

$$\begin{aligned}\varphi(x)|0\rangle&\rightarrow\int\frac{d^{3}p}{(2\pi)^{3}}\frac{1}{2E_{p}}e^{ipx}|\Lambda p\rangle\overset{p\rightarrow\Lambda^{-1}p}{=}\int\frac{d^{3}\Lambda^{-1}p}{(2\pi)^{3}}\frac{1}{2E_{\Lambda^{-1}p}}e^{i\left(\Lambda^{-1}p\right)x}\left|\Lambda\Lambda^{-1}p\right\rangle\\&=\int\frac{d^{3}p}{(2\pi)^{3}}\frac{1}{2E_{p}}e^{i\left(\Lambda^{-1}p\right)\left(\Lambda^{-1}\Lambda x\right)}|p\rangle=\int\frac{d^{3}p}{(2\pi)^{3}}\frac{1}{2E_{p}}e^{ip(\Lambda x)}|p\rangle=\varphi(\Lambda x)|0\rangle\end{aligned}$$

so this object is covariant in a well defined sense. On the other hand, the state $\varphi(x)|0\rangle$ is well localized, since

$$\langle0|a(x')\varphi(x)|0\rangle=\langle0|\int\frac{d^{3}p}{(2\pi)^{3}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}e^{ip(x-x')}|0\rangle$$

and this integral decreases rapidly for $|\vec{x}-\vec{x}'|$ greater than the Compton wavelength of the particle $\hbar/mc$, i.e.$1/m$. (Exercise: convince yourself about this. Hint: use Mathematica or something similar.)

Conclusion: $\varphi(x)|0\rangle$ is a reasonable relativistic generalization of a state of a localized particle. Together with the rest of the world, we will treat $\varphi(x)|0\rangle$ as a handy compromise between covariance and localizability.

[20] We are trying to avoid the natural notation $a^{+}(x)|0\rangle=|x\rangle$ here, since the symbol $|x\rangle$ is reserved for a different quantity in Peskin-Schroeder. Anyway, we want to use it at least in this footnote, to stress that in this notation $|x\rangle\nrightarrow|\Lambda x\rangle$ in spite of what intuition may suggest. This fact emphasizes a need for proof of the transformation $|p\rangle\nrightarrow|\Lambda p\rangle$ which is intuitively equally "clear".

2.2.2 Contemplations and subtleties

Let us summarize our achievements: we have undergone a relatively exhaustive journey to come to almost obvious results. The (relativistic quantum) theory (of free particles) is formulated in the Fock space, which is something to be expected from the very beginning. The basis vectors of this space transform in the natural way. Hamiltonian of the system of free particles is nothing else but the well known beast, found easily long ago (see Introductions).

Was all this worth the effort, if the outcome is something we could guess with almost no labor at all? Does one get anything new? One new thing is that now we have not only the Hamiltonian, but all 10 Poincarè generators — this is the "leitmotiv" of our development of the QFT. All generators are expressible in terms of $a^+_{\vec{p}}(t)$ and $a_{\vec{p}}(t)$ but, frankly, for the free particles this is also relatively straightforward to guess.

The real yield of the whole procedure remains unclear until one proceeds to the interacting fields or particles. The point is that, even if being motivated by the free field case, the Fourier expansion of fields and quantization in terms of the Fourier coefficients turns out to be an efficient tool also for interacting fields. Even in this case the canonical quantization provides the 10 Poincarè generators in terms of the fields $\varphi(\vec{x},t)$, i.e. in terms of $a^+_{\vec{p}}(t)$ and $a_{\vec{p}}(t)$, which again have (in a sense) a physical meaning of creation and annihilation operators.

Unfortunately, all this does not go smoothly. In spite of our effort to pretend the opposite, the canonical quantization of systems with infinitely many DOF is much more complex than of those with a finite number of DOF. The only reason why we were not faced with this fact hitherto, is that for the free fields the difficulties are not inevitably manifest. More precisely, there is a representation (one among infinitely many) of canonical commutation relations which looks almost like if the system has a finite number of DOF. Not surprisingly, this is the Fock representation — the only one discussed so far. For interacting fields, however, the Fock space is not the trouble-free choice any more. In this case neither the Fock space, nor any other explicitly known representation, succeeds in avoiding serious difficulties brought in by the infinite number of DOF.

In order to understand, at least to some extent, the problems with the quantization of interacting fields, the said difficulties are perhaps worth discussion already for the free fields. So are the reasons why these difficulties are not so serious in the free field case.

Let us start with recollections of some important properties of systems defined by a finite number of canonical commutation relations $[p_i,q_j]=-i\delta_{ij}$ and $[p_i,p_j]=[q_i,q_j]=0$, where $i,j=1,\ldots,n$. One can always introduce operators $a_i=q_ic_i/2+ip_i/c_i$ and $a_i^+=q_ic_i/2-ip_i/c_i$ where c_i is a constant (for harmonic oscillators the most convenient choice is $c_i=\sqrt{2m_i\omega_i}$) satisfying $\left[a_i,a_j^+\right]=\delta_{ij}$ and $[a_i,a_j]=\left[a_i^+,a_j^+\right]=0$. The following holds:

- a state $|0\rangle$ annihilated by all a_i operators does exist $(\exists\,|0\rangle\,\forall i a_i\,|0\rangle=0)$
- the Fock representation of the canonical commutation relations does exist
- all irreducible representations are unitary equivalent to the Fock one
- Hamiltonians and other operators are usually well-defined

For infinite number of DOF, i.e. for the same set of commutation relations, but with $i,j=1,\ldots,\infty$ the situation is dramatically different:

- existence of $|0\rangle$ annihilated by all a_i operators is not guaranteed
- the Fock representation, nevertheless, does exist

- there are infinitely many representations non-equivalent to the Fock one
- Hamiltonians and other operators are usually ill-defined in the Fock space

Let us discuss these four point in some detail.

As to the existence of $|0\rangle$, for one oscillator the proof is notoriously known from QM courses. It is based on well-known properties of the operator $N = a^+a$: 1. $N|n\rangle = n|n\rangle \Rightarrow Na|n\rangle = (n-1)a|n\rangle$ $(a^+aa = [a^+, a]\,a + aa^+a = -a + aN)$
2. $0 \leq \|a|n\rangle\|^2 = \langle n|a^+a|n\rangle = n\langle n|n\rangle$ implying $n \geq 0$, which contradicts 1 unless the set of eigenvalues n contains 0. The corresponding eigenstate is $|0\rangle$.

For a finite number of independent oscillators the existence of the *common* $|0\rangle$ $(\forall i\ a_i|0\rangle = 0)$ is proven along the same lines. One considers the set of commuting operators $N_i = a_i^+ a_i$ and their sum $N = \sum_i a_i^+ a_i$. The proof is basically the same as for one oscillator.

For infinite number of oscillators, however, neither of these two approaches (nor anything else) really works. The step by step argument proves the statement for any finite subset of N_i, but fails to prove it for the whole infinite set. The proof based on the operator N refuses to work once the convergence of the infinite series is discussed with a proper care.

Instead of studying subtleties of the breakdown of the proofs when passing from finite to infinite number of oscillators, we will demonstrate the existence of the so-called strange representations of a_i^+, a_i (representations for which there is no vacuum state $|0\rangle$) by an explicit construction (Haag 1955). Let a_i^+, a_i be the creation and annihilation operators in the Fock space with the vacuum state $|0\rangle$. Introduce their linear combinations $b_i = a_i \cosh\alpha + a_i^+ \sinh\alpha$ and $b_i^+ = a_i \sinh\alpha + a_i^+ \cosh\alpha$. Commutation relations for the b-operators are the same as for the a-operators (check it). Now let us assume the existence of a state vector $|0_\alpha\rangle$ satisfying $\forall i$ $b_i|0_\alpha\rangle = 0$. For such a state one would have

$$0 = \langle i|b_j|0_\alpha\rangle = \langle i,j|0_\alpha\rangle \cosh\alpha + \langle 0|0_\alpha\rangle\,\delta_{ij}\sinh\alpha$$

which implies $\langle i,i|0_\alpha\rangle = \text{const}$ (no i dependence). Now for i being an element of an *infinite* index set this constant must vanish, because otherwise the norm of the $|0_\alpha\rangle$ state comes out infinite $(\langle 0_\alpha|0_\alpha\rangle \geq \sum_{i=1}^\infty |\langle i,i|0_\alpha\rangle|^2 = \sum_{i=1}^\infty \text{const}^2)$. And since $\text{const} = -\langle 0|0_\alpha\rangle \tanh\alpha$ the zero value of this constant implies $\langle 0|0_\alpha\rangle = 0$. Moreover, vanishing $\langle 0|0_\alpha\rangle$ implies $\langle i,j|0_\alpha\rangle = 0$.

It is straightforward to show that also $\langle i|0_\alpha\rangle = 0$ $(0 = \langle 0|b_i|0_\alpha\rangle = \langle i|0_\alpha\rangle\cosh\alpha)$ and finally $\langle i,j,\ldots|0_\alpha\rangle = 0$ by induction

$$\underbrace{\langle i,j,\ldots|}_{n} b_k|0_\alpha\rangle = \underbrace{\langle i,j,\ldots|}_{n+1} 0_\alpha\rangle\cosh\alpha + \underbrace{\langle i,j,\ldots|}_{n-1} 0_\alpha\rangle\sinh\alpha$$

But $\langle i,j,\ldots|$ form a basis of the Fock space, so we can conclude that within the Fock space there is no vacuum state, i.e. a non-zero normalized vector $|0_\alpha\rangle$ satisfying $\forall i$ $b_i|0_\alpha\rangle = 0$.

Representations of the canonical commutation relations without the vacuum vector are called *the strange representations*. The above example[21] shows not only that such representations exist,

[21] Another instructive example (Haag 1955) is provided directly by the free field. Here the standard annihilation operators are given by $a_{\vec p} = \varphi(\vec p, 0)\sqrt{\omega_{\vec p}/2} + i\pi(\vec p, 0)/\sqrt{2\omega_{\vec p}}$, where $\omega_{\vec p} = \vec p^{\,2} + m^2$. But one can define another set $a'_{\vec p}$ in the same way, just with m replaced by some $m' \neq m$. Relations between the two sets are (check it) $a'_{\vec p} = c_+ a_{\vec p} + c_- a^+_{-\vec p}$ and $a'^+_{\vec p} = c_- a^+_{\vec p} + c_+ a_{-\vec p}$, where $2c_\pm = \sqrt{\omega'_{\vec p}/\omega_{\vec p}} \pm \sqrt{\omega_{\vec p}/\omega'_{\vec p}}$. The commutation relations become $\left[a'_{\vec p}, a'^+_{\vec k}\right] = (2\pi)^3\,\delta(\vec p - \vec k)(\omega'_{\vec p} - \omega_{\vec p})/2\omega'_{\vec p}$ and $\left[a'_{\vec p}, a'_{\vec k}\right] = \left[a'^+_{\vec p}, a'^+_{\vec k}\right] = 0$. The rescaled operators $b_{\vec p} = r_{\vec p} a'_{\vec p}$ and $b^+_{\vec p} = r_{\vec p} a^{+\prime}_{\vec p}$ where $r^2_{\vec p} = 2\omega'_{\vec p}/(\omega'_{\vec p} - \omega_{\vec p})$ constitutes a representation of the canonical commutation relations.

If there is a vacuum vector for a-operators, i.e. if $\exists\,|0\rangle\ \forall \vec p\ a_{\vec p}|0\rangle = 0$, then there is no $|0'\rangle$ satisfying $\forall \vec p\ b_{\vec p}|0'\rangle = 0$

but that one can obtain (some of) them from the Fock representation by very simple algebraic manipulations.

As to the Fock representation, it is always available. One just has to postulate the existence of the vacuum $|0\rangle$ and then to build the basis of the Fock space by repeated action of a_i^+ on $|0\rangle$. Let us emphasize that even if we have proven that existence of such a state does not follow from the commutation relations in case of infinite many DOF, we are nevertheless free to postulate its existence and investigate the consequences. The very construction of the Fock space guarantees that the canonical commutation relations are fulfilled.

Now to the (non-)equivalence of representations. Let us consider two representations of canonical commutation relations, i.e. two sets of operators a_i, a_i^+ and $a'_i, a_i'^+$ in Hilbert spaces $\mathbb{H}$ and $\mathbb{H}'$ correspondingly. The representations are said to be equivalent if there is an unitary mapping $\mathbb{H} \stackrel{U}{\to} \mathbb{H}'$ satisfying $a'_i = Ua_iU^{-1}$ and $a_i'^+ = Ua_i^+U^{-1}$.

It is quite clear that the Fock representation cannot be equivalent to a strange one. Indeed, if the representations are equivalent and the non-primed one is the Fock representation, then defining $|0'\rangle = U|0\rangle$ one has $\forall i\ a'_i|0'\rangle = Ua_iU^{-1}U|0\rangle = Ua_i|0\rangle = 0$, i.e. there is a vacuum vector in the primed representation, which cannot be therefore a strange one.

Perhaps less obvious is the fact that as to the canonical commutation relations, any *irreducible representation* (no invariant subspaces) *with the vacuum vector* is equivalent to the Fock representation. The proof is constructive. The considered space $\mathbb{H}'$ contains a subspace $\mathbb{H}_1 \subset \mathbb{H}'$ spanned by the basis $|0'\rangle$, $a_i'^+|0'\rangle$, $a_i'^+a_j'^+|0'\rangle$, ... One defines a linear mapping U from the subspace $\mathbb{H}_1$ on the Fock space $\mathbb{H}$ as follows: $U|0'\rangle = |0\rangle$, $Ua_i^{+\prime}|0'\rangle = a_i^+|0\rangle$, $Ua_i'^+a_j'^+|0'\rangle = a_i^+a_j^+|0\rangle$, ... The mapping U is clearly invertible and preserves the scalar product, which implies unitarity (Wigner's theorem). It is also straightforward that operators are transformed as $Ua_i'^+U^{-1} = a_i^+$ and $Ua'_iU^{-1} = a_i$. The only missing piece is to show that $\mathbb{H}_1 = \mathbb{H}'$ and this follows, not surprisingly, form the irreducibility assumption[22].

An immediate corollary of the above considerations is that all irreducible representations of the canonical commutation relations for finite number of DOF are equivalent (Stone–von Neumann). Indeed, having a finite number of DOF they are obliged to have a vacuum state, and having a vacuum state they are necessarily equivalent. As to the (non-)equivalence of various strange representations, we are not going to discuss the subject here. Let us just remark that a complete classification of strange representations of the canonical commutation relations is not known yet.

Before going further, we should mention an important example of a reducible representation with a vacuum state. Let us consider perhaps the most natural (at least at the first sight) representation of a quantized system with infinite many DOF — the one in which a state is represented by a function $\psi(q_1, q_2, \ldots)$ of infinitely many variables[23]. The function $\psi_0(q_1, q_2, \ldots) = \prod_{i=1}^{\infty} \varphi_0(q_i)$, where φ_0 is a wavefunction of the ground state of LHO, is killed by all annihilation

(the proof is the same as in the example in the main text). In other words at least one of the representations under consideration is a strange one.

Yet another example is provided by an extremely simple prescription $b_{\vec{p}} = a_{\vec{p}} + \alpha(\vec{p})$, where $\alpha(\vec{p})$ is a complex-valued function. For $\int |\alpha(\vec{p})|^2 = \infty$ this representation is a strange one (the proof is left to the reader as an exercise)

[22] As always with this types of proofs, if one is not quite explicit about definition domains of operators, the "proof" is a hint at best. For the real, but still not complicated, proof along the described lines see Berezin, Metod vtornicnovo kvantovania, p.24.

[23] Of course, not any such function can represent a state. Recall that for one variable, only functions from L^2 qualify for states. To proceed systematically, one has to define a scalar product, which can be done for specific functions of the form $\psi(q_1, q_2, \ldots) = \prod_{i=1}^{\infty} \psi_i(q_i)$ in a simple way as $\psi.\psi' = \prod_{i=1}^{\infty} \int dq_i \psi_i^*(q_i)\psi'_i(q_i)$. This definition can be extended to the linear envelope of the "quadratically integrable specific functions" and the

operators, so it represents the vacuum state. Nevertheless, the Hilbert space $\mathbb{H}$ of such functions cannot be unitary mapped on the Fock space $\mathbb{H}_B$, because of different dimensionalities (as already discussed, $\mathbb{H}$ is non-separable, while $\mathbb{H}_B$ is separable). The Fock space can be constructed from this vacuum, of course, and it happens to be a subspace of $\mathbb{H}$ (invariant with respect to creation and annihilation operators). This Fock space, however, does not cover the whole $\mathbb{H}$. What is missing are states with actually infinite number of particles. The point is that only states with finite, although arbitrarily large, number of particles are accessible by repeated action of the creator operators on the vacuum vector[24].

This brings us back to the question of how does it come that for infinitely many oscillators we got a separable, rather then a non-separable, Hilbert space. It should be clear now that this is just a matter of choice — the Fock space is not the only option, we could have chosen a non-separable Hilbert space (or a separable strange representation) as well. The main advantage of the Fock space is that the relevant mathematics is known. On the other hand, the Fock space also seems to be physically acceptable, as far as all physically relevant states do not contain infinitely many particles. It would be therefore everything but wise to ignore the fact that thanks to the Fock space we can proceed further without a development of a new and difficult mathematics. So we will, like everybody does, try to stick to this fortunate encounter and milk it as much as possible.

Anyway, the choice of the Fock space as the playground for QFT does not close the discussion. It may turn out that the Hamiltonian and other Poincarè generators are ill-defined in the Fock space. For the free field, fortunately, the generators turn out to be well defined. But the reader should make no mistake, this is an exception rather than a rule.

The last point from the above lists of characteristic features of systems of finite and infinite DOF concern definitions of operators. This is a subtle point already for a finite number of DOF[25]. For systems with an infinite number of DOF the situation is usually even worse. The reason is that many "natural" operators are of the form O^n where $O = \sum_{i=1}^{\infty} c_i a_i^+ + c_i^* a_i$. The trouble now is that for *infinite* sum one can have $\sum_{i=1}^{\infty} |c_i|^2 = \infty$, the quantum field $\varphi(x)$ is a prominent example. Such an operator leads to a state with an infinite norm acting on any standard basis vector in the Fock space (convince yourself). But this simply means that such operators are not defined within the Fock space.

Nevertheless, the operator O, as well as the quantum field $\varphi(x)$, has a finite matrix elements between any two standard basis vectors. This enables us to treat them as objects having not a well defined meaning as they stand, but only within scalar products — a philosophy similar to

space of normalized functions is to be checked for completeness. But as to the mathematical rigor, this remark represents the utmost edge of our exposition.

[24]This may come as a kind of surprise, since due to the infinite direct sum $\oplus_{n=0}^{\infty}\mathbb{H}^n$ in the definition of the Fock space, one may expect (incorrectly) that it also contains something like $\mathbb{H}^{\infty}$. This symbol, however, is just an abuse of notation — it does not correspond to any many-particle subspace of the Fock space. An analogy may be of some help in clarifying this issue: the set of natural numbers $\mathbb{N}$ does not contain an infinite number ∞, even if it contains every n where $n = 1, \ldots, \infty$.

[25]The point is that (unbounded) operators in quantum theory usually enter the game in the so-called formal way, i.e. without a precise specification of domains. Precise domains, on the other hand, are of vital importance for such attributes as selfadjointness, which in turn is a necessary condition for a Hamiltonian to define a dynamics, i.e. a unitary time evolution (Stone theorem). Formal Hamiltonians are usually Hermitian (symmetric) on a dense domain in the Hilbert space, and for some (but not for all) such symmetric operators the selfadjoint extensions do exist. If so, the Hamiltonian is considered to be well-defined.

For our present purposes the important thing is that the Hamiltonian of the LHO is well-defined in this strict sense. One can even show, using sophisticated techniques of modern mathematical physics, that the Hamiltonian of an anharmonic oscillator $H = p^2/2m + q^2 + q^4$ is well defined (see e.g. Reed-Simon, volume 2, for five proofs of this statement) and this holds for any finite number of oscillators. On the other hand, some formal Hamiltonians are doomed to be ill-defined and lead to no dynamics whatsoever.

that of distributions like the δ-function. Operators which can be defined only in this sense are sometimes called improper operators.

But the main problem is yet to come. Were all operators appearing in QFT proper or improper ones, the QFT would be perhaps much easier and better understood then it actually is. Unfortunately, for many "natural" operators even the scalar products are infinite. Such objects are neither proper, nor improper operators, they are simply senseless expressions.

Nevertheless, the free field Hamiltonian $H = \int d^3p \; a^+_{\vec{p}}\,(t)\, a_{\vec{p}}\,(t)\, \omega_{\vec{p}} / \,(2\pi)^3$ is a proper (even if unbounded) operator in the Fock space, since it maps an n-particle basis state to itself, multiplied by a finite number[26]. The other generators map n-particle basis states to normalized n-particle states, so all these operators are well defined. That is why all difficulties discussed in this section remain hidden in the free field case. But they will reappear quickly, once the interacting fields are considered.

[26]The remaining question is if this unbounded Hamiltonian has a selfadjoint extension. The answer is affirmative, the proof, however, is to be looked for in books on modern mathematical physics rather than in introductory texts on QFT. One may raise an objection that we have demonstrated selfadjointness of the free field Hamiltonian indirectly by finding the explicit unitary time evolution of states (which follows from the time evolution of the creation and annihilation operators). This, however, was found by formal manipulations, without bothering about if the manipulated objects are well defined. Needless to say, such an approach can lead to contradictions. Anyway, for the free fields the formal manipulations are fully supported by more careful analysis.

All this applies, of course, only for one particular ordering of creation and annihilation operators — not surprisingly the one we have adopted. Other orderings are, strictly speaking, the above mentioned senseless expressions with infinite matrix elements between basis vectors.

Chapter 3

Interacting Quantum Fields

3.1 Naive approach

In the last section of the previous chapter we have discussed several unpleasant features which may appear in a theory of interacting fields (strange representations, ill-defined operators, no dynamics in a sense of unitary time evolution). In the first two sections of the present chapter we are going to ignore all this completely. On top of that, in the first section we will oversimplify the matters even more than is the common habit.

The reason for this oversimplification is purely didactic. As we will see, one can get pretty far using a bit simple-minded approach and almost everything developed in this framework will survive, with necessary modifications, later critical reexamination. The said modifications are, on the other hand, quite sophisticated and both technically and conceptually demanding. We prefer, therefore, to postpone their discussion until the basic machinery of dealing with interacting fields is developed in the simplified naive version[1].

As the matter of fact, the naive approach is the most natural one. It is based on the assumption that the free field lagrangian defines what particles are[2], and the interaction lagrangian defines how do these particles interact with each other. Life, however, turns out to be surprisingly more complex.

So it happens that by switching on the interaction, one in fact redefines what particles are. This rather non-trivial and surprising fact has to be taken into account — otherwise one is, sooner or later, faced with serious inconsistencies in the theory. In the standard approach one indeed develops the theory of interacting quantum fields having in mind from the very beginning that "particle content" of the free and interacting theories may differ significantly.

In our naive approach we will ignore all this and move on happily until we will understand almost completely where the Feynman rules come from. The few missing ingredients will be obtained afterwards within the standard approach.

[1] It should be stressed that even after all known modifications (see section 3.2) the resulting theory of interacting quantum fields is still not satisfactory in many respects (see section **??**). The difference between the oversimplified and the standard approach is not that they are incorrect and correct respectively, but rather that they are incorrect to different degrees.

[2] According to the naive approacg, one-particle states are those obtained by acting of the creation operator $a^+_{\vec{p}}$ on the vacuum state $|0\rangle$, two-particle states are those obtained by acting of two creation operators on the vacuum state, etc.

canonical quantization of interacting fields

For interacting fields the quantization proceeds basically along the same lines as for the free fields. A particular theory is defined by a Lagrangian density in the form of a sum of the free and the interaction Lagrangian densities. For relativistic theory this Lagrangian density is a Lorentz (as well as Poincaré) scalar. Invariance of the Lagrangian density with respect to Poincaré transformations provides us with 10 Noether's charges.
Example: *φ^4-theory (we obtained the following results already in 2.1.1)*

$$\begin{aligned}
\textit{t-translations} \qquad Q^0 &= \frac{1}{2}\int d^3x\ (\dot{\varphi}^2 + |\nabla\varphi|^2 + m^2\varphi^2) \\
&\quad + \frac{1}{2}\int d^3x\ \frac{1}{12} g\varphi^4 \\
\textit{$\vec{x}$-translations} \qquad \vec{Q} &= \int d^3x\ \dot{\varphi}\ \nabla\varphi \\
\textit{rotations} \qquad \vec{Q}_R &= -\int d^3x\ \dot{\varphi}\ \vec{x}\times\nabla\varphi \\
\textit{boosts} \qquad \vec{Q}_B &= -t\int d^3x\ \dot{\varphi}\ \nabla\varphi + \frac{1}{2}\int d^3x\ \vec{x}\ (\dot{\varphi}^2 + |\nabla\varphi|^2 + m^2\varphi^2) \\
&\quad + \frac{1}{2}\int d^3x\ \vec{x}\ \frac{1}{12} g\varphi^4
\end{aligned}$$

After Hamiltonian reformulation of the theory, the Poisson algebra of these charges is isomorphic to the Lie algebra of the corresponding Poincaré generators. And after subsequent quantization, this Poisson algebra leads to the Lie algebra of the quantized charges, which defines a relativistic quantum theory (i.e. a representation of the Poincaré group within a Hilbert space).

Example: *φ^4-theory*

$$\begin{gathered}
L = \textstyle\int d^3x\ \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}g\varphi^4 \\
\downarrow \\
H = \textstyle\int d^3x\ \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{4!}g\varphi^4 \\
\{\pi(\vec{x},t), \varphi(\vec{y},t)\} = \delta^3(\vec{x}-\vec{y}) \\
\{\pi(\vec{x},t), \pi(\vec{y},t)\} = \{\varphi(\vec{x},t), \varphi(\vec{y},t)\} = 0 \\
\frac{dF[\varphi,\pi]}{dt} = \{H, F\} \\
\downdownarrows \\
\hat{H} = \textstyle\int d^3x\ \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}|\nabla\hat{\varphi}|^2 + \frac{1}{2}m^2\hat{\varphi}^2 + \frac{1}{4!}g\hat{\varphi}^4 \\
[\hat{\pi}(\vec{x},t), \hat{\varphi}(\vec{y},t)] = -i\hbar\delta^3(\vec{x}-\vec{y}) \\
[\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{y},t)] = [\hat{\varphi}(\vec{x},t), \hat{\varphi}(\vec{y},t)] = 0 \\
\frac{dF[\hat{\varphi},\hat{\pi}]}{dt} = \frac{i}{\hbar}\left[\hat{H}, F\right] \\
\downarrow \\
\mathbb{H} = ???
\end{gathered}$$

As in the case of free fields, the problem now is that natural choice of $\mathbb{H}$ is a non-separable Hilbert space. For free fields we managed to avoid the problem by taking a strange detour route, i.e. by rewriting the scalar field in terms of harmonic oscilators. This lead us to a Fock subspace

of the non-separable space, which turned out to be a sufficient playground (when starting form this subspace, one was never expelled from it by the free field dynamics). It is quite natural to try the same trick also for interacting fields.

So the next step is the Fourier expansion of the classical fields and the conjugated momenta

$$\varphi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}(t)\, e^{i\vec{p}.\vec{x}} + a^+_{\vec{p}}(t)\, e^{-i\vec{p}.\vec{x}} \right)$$
$$\pi(\vec{x},t) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}}(t)\, e^{i\vec{p}.\vec{x}} - a^+_{\vec{p}}(t)\, e^{-i\vec{p}.\vec{x}} \right)$$

which leads (after quatization) to the commutation relations

$$[a_{\vec{p}}(t), a^+_{\vec{p}'}(t)] = (2\pi)^3 \delta(\vec{p} - \vec{p}') \qquad [a_{\vec{p}}(t), a_{\vec{p}'}(t)] = [a^+_{\vec{p}}(t), a^+_{\vec{p}'}(t)] = 0$$

Let us emphasize that the commutation relations hold for arbitrary time t which, however, must be the same for both operators in the commutator — that is why they are known as "equal-time commutation relations". At any fixed time, these commutation relations can be represented by creation and annihilation operators in the Fock space.

So far, it looks like there is no serious difference between quantization of free fields and the interacting ones. Poincaré generators are, as a rule, more complicated for interacting fields, but otherwise the whole procedure looks pretty similar in both cases. For the free fields, appearence of the Fock space was the last important step which enabled us to complete the canonical quantization program. For the interacting fields, however, one does not have the Fock space, but rather Fock spaces.

The point is that for different times t the $a^+_{\vec{p}}(t)$ and $a_{\vec{p}'}(t)$ operators are, in principle, represented in different Fock subspaces of the "large" non-separable space. For the free fields all these Fock spaces coincide, they are in fact just one Fock space — we were able to demonstrate this due to the explicit knowledge of the time evolution of $a^+_{\vec{p}}$ and $a_{\vec{p}'}$. For interacting fields, however, such a knowledge is not at our disposal anymore. One of the main differences between the free and interacting fields is that the time evolution becomes highly nontrivial for the latter. In the Heisenberg picture, the equations for $a_{\vec{p}}(t)$ and $a^+_{\vec{p}'}(t)$ do not lead to simple harmonic time-dependence, nor do the equations for the basis states in the Schrödinger picture (let us epmhasize that basis vectors are eigenstates of the free, rather than the full Hamiltonian).

One of the consequences of the non-trivial and unknown time dependence of the interacting fields is that, frankly speaking, we do not understand our playground. For interacting fields the Fock spaces defined by $a^+_{\vec{p}}(t)$ and $a_{\vec{p}'}(t)$ at different times cannot be proven to coincide. And even if they did, we do not know the representation of the Poincaré algebra explicitly. The generators are defined in terms of $a^+_{\vec{p}}(t)$ and $a_{\vec{p}'}(t)$ and these operators are explicitly known only at one specific moment (let's say at $t = 0$)

How to proceed further in such circumstances? It is a common habit in quantum field theory to ignore the difficulties related to non-separability as long as possible (which is usualy rather long indeed, for most of us it is simply life-long ignorance). The said difficulties are evaded (to a certain degree) by a clever approximative scheme, namely by the perturbation theory in the so-called interaction picture. In this section, we will develop the scheme and learn how to use it in the simplified version. The scheme is valid also in the standard approach, but its usage is a bit different (as will be discussed thoroughly in the next section).

3.1.1 Interaction picture

Our main aim will be the development of some (approximate) techniques of solving the time evolution of interacting fields in the interaction picture of the time evolution in QFT. Operators and states in the interaction picture are defined as[3]

$$A_I(t) = e^{iH_0t}e^{-iHt}A_H(t)\,e^{iHt}e^{-iH_0t}$$
$$|\psi_I(t)\rangle = e^{iH_0t}e^{-iHt}\,|\psi_H\rangle$$

where the operators H and H_0 are understood in the Schrödinger picture.

The time evolution of operators in the interaction picture is quite simple, it is equal to the time evolution of the free fields. Indeed, both time evolutions (the one of the free fields and the one of the interacting fields in the interaction picture) are controlled by the same Hamiltonian H_0.

Let us emphasize the similarities and the differences between the interacting fields in the Heisenberg and the interaction pictures. In both pictures one has identically looking expansions

$$\varphi_H(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p},H}(t) + a^+_{-\vec{p},H}(t)\right)e^{i\vec{p}.\vec{x}}$$
$$\varphi_I(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p},I}(t) + a^+_{-\vec{p},I}(t)\right)e^{i\vec{p}.\vec{x}}$$

However, the explicit time dependence of the creation and annihilation operators in the Heisenberg picture is unknown, while in the interaction picture it is known explicitly as $a^+_{\vec{p},I}(t) = a^+_{\vec{p}}e^{i\omega_{\vec{p}}t}$ and $a_{\vec{p},I}(t) = a_{\vec{p}}e^{-i\omega_{\vec{p}}t}$ (see section**??**). Using the free field results, one can therefore write immediately

$$\varphi_I(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}e^{-ipx} + a^+_{\vec{p}}e^{ipx}\right)$$

where $a^+_{\vec{p}}$ and $a_{\vec{p}}$ are the creation and annihilation operators at $t=0$ (in any picture, they all coincide at this moment). The explicit knowledge and the space-time structure (scalar products of 4-vectors) of the φ_I-fields are going to play an extremely important role later on.

The time evolution of states in the interaction picture is given by

$$i\partial_t\,|\psi_I\rangle = H_I(t)\,|\psi_I\rangle \qquad H_I(t) = e^{iH_0t}\,(H-H_0)\,e^{-iH_0t}$$

where H and H_0 are understood in the Schrödinger picture. The operator $H_I(t)$ is *the interaction Hamiltonian in the interaction picture.*

Needless to say, solving the evolution equation for states in the interaction picture is the difficult point. Nevertheless, we will be able to give the solution as a perturbative series in terms of $\varphi_I(x)$. To achieve this, however, we will need to express all quantities, starting with $H_I(t)$, in terms of $\varphi_I(x)$.

[3]Relations between the Schrödinger, Heisenberg and interaction pictures:

$$A_H(t) = e^{iHt}A_Se^{-iHt} \qquad |\psi_H\rangle = e^{iHt}\,|\psi_S(t)\rangle$$
$$A_I(t) = e^{iH_0t}A_Se^{-iH_0t} \qquad |\psi_I(t)\rangle = e^{iH_0t}\,|\psi_S(t)\rangle$$

The operators H and H_0 are understood in the Schrödinger picture. Their subscripts are omitted for mainly esthetic reasons (to avoid too much make-up in the formulae). Anyway, directly from the definitions one has $H_H = H_S$ and $H_{0,I} = H_{0,S}$, therefore the discussed subscripts would be usually redundant.

The canonical quantization provides the Hamiltonian as a function of fields in the Heisenberg picture. What we will need is $H_I(t)$ expressed in terms of $\varphi_I(x)$. Fortunately, this is straightforward: one just replaces φ_H and π_H operators in the Heisenberg picture by these very operators in the interaction picture, i.e. by φ_I and π_I. Proof: one takes $t=0$ in the Heisenberg picture, and in thus obtained Schrödinger picture one simply inserts $e^{-iH_0t}e^{iH_0t}$ between any fields or conjugate momenta[4].

Example: *φ^4-theory* $\quad \mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}g\varphi^4$

$$H = \int d^3x(\frac{1}{2}\pi_H^2 + \frac{1}{2}|\nabla\varphi_H|^2 + \frac{1}{2}m^2\varphi_H^2 + \frac{1}{4!}g\varphi_H^4)$$

and taking $t=0$ one gets $H_{\rm int} = \int d^3x\ \frac{1}{4!}g\varphi_S^4$, leading to

$$H_I = \int d^3x\ \frac{1}{4!}g\varphi_I^4$$

In what follows, the key role is going to be played by the operator $U(t,t')$, which describes the time evolution of states *in the interaction picture*

$$|\psi_I(t)\rangle = U(t,t')\,|\psi_I(t')\rangle$$

Directly from the definition one has

$$U(t,t'') = U(t,t')\,U(t',t'') \qquad\qquad U^{-1}(t,t') = U(t',t)$$

where the second relation follows from the first one and the obvious identity $U(t,t)=1$. Differentiating with respect to t one obtains $i\partial_t U(t,t')\,|\psi_I(t')\rangle = H_I(t)\,U(t,t')\,|\psi_I(t')\rangle$ for every $|\psi_I(t')\rangle$ and therefore

$$i\partial_t U(t,t') = H_I(t)\,U(t,t')$$

with the initial condition $U(t,t)=1$.

For $t'=0$ the solution of this equation is readily available[5]

$$U(t,0) = e^{iH_0t}e^{-iHt}$$

(H_0 and H in the Schrödinger picture). This particular solution shows that (in addition to providing the time evolution in the interaction picture) the $U(t,0)$ operator enters the relation between the field operators in the Heisenberg and interaction pictures[6]

$$\varphi_H(x) = U^{-1}\left(x^0,0\right)\varphi_I(x)\,U\left(x^0,0\right)$$

[4]Remark: the simple replacement $\varphi_H \to \varphi_I$, $\pi_H \to \pi_I$ works even for gradients of fields, one simply has to realize that $e^{-iH_0t}\nabla\varphi_S e^{iH_0t} = \nabla\left(e^{-iH_0t}\varphi_S e^{iH_0t}\right) = \nabla\varphi_I$, which holds because H_0 does not depend on the space coordinates.

[5]Indeed $\partial_t e^{iH_0t}e^{-iHt} = e^{iH_0t}(iH_0 - iH)e^{-iHt} = -ie^{iH_0t}H_{\rm int}e^{-iH_0t}e^{iH_0t}e^{-iHt} =$
$= -iH_I(t)e^{iH_0t}e^{-iHt}$. Note that the very last equality requires $t'=0$ and therefore one cannot generalize the relation to any t'. In general $U(t,t') \neq e^{iH_0(t-t')}e^{-iH(t-t')}$.

[6]$A_H = e^{iHt}e^{-iH_0t}A_I e^{iH_0t}e^{-iHt} = U^{-1}(t,0)A_I U(t,0)$

3.1.2 Transition amplitudes

The dynamical content of a quantum theory is encoded in transition amplitudes, i.e. the probability amplitudes for the system to evolve from an initial state $|\psi_i\rangle$ at t_i to a given state $|\psi_f\rangle$ at t_f. These probability amplitudes are coefficients of the expansion of the final state (evolved from the given initial state) in the basis defined by vectors $|\psi_f\rangle$.

In the Schrödinger picture the initial and the final states are $|\psi_i\rangle$ and $U_S(t_f,t_i)|\psi_i\rangle$ respectively, where $U_S(t_f,t_i)=\exp\{-iH(t_f-t_i)\}$ is the time evolution operator in the Schrödinger picture. So in this picture one has

$$\text{transition amplitude} = \langle\psi_f| U_S(t_f,t_i)|\psi_i\rangle$$

It should be perhaps stressed that, in spite of what the notation might suggest, $|\psi_f\rangle$ does not define the final state (which is rather defined by $|\psi_i\rangle$ and the time evolution). Actually $|\psi_f\rangle$ just defines what component of the final state we are interested in.

In the Heisenberg picture, the time evolution of states is absent. Nevertheless, the transition amplitude can be easily written in this picture as[7]

$$\text{transition amplitude} = \langle\psi_{f,H}|\psi_{i,H}\rangle$$

where $|\psi_{i,H}\rangle = e^{iHt_i}|\psi_S(t_i)\rangle = e^{iHt_i}|\psi_i\rangle$ and $|\psi_{f,H}\rangle = e^{iHt_f}|\psi_S(t_f)\rangle = e^{iHt_f}|\psi_f\rangle$.

And last, but not least, in the interaction picture, one has[8]

$$\text{transition amplitude} = \langle\psi_{f,I}| U(t_f,t_i)|\psi_{i,I}\rangle$$

where $|\psi_{i,I}\rangle = e^{iH_0t_i}|\psi_S(t_i)\rangle = e^{iH_0t_i}|\psi_i\rangle$ and $|\psi_{f,I}\rangle = e^{iH_0t_f}|\psi_S(t_f)\rangle = e^{iH_0t_f}|\psi_f\rangle$. Note that the index I for the time evolution operator $U(t,t')$ is omitted. Throughout this text $U(t,t')$ always means the time evolution operator of states in the interaction picture.

In what follows we are going to encounter slightly generalized objects, namely transition amplitudes with the time evolution interupted by action of particular operators at particular times. Let us consider two such operators A_1 and A_2 acting at times t_1 and t_2 respectively (generalization to arbitrary number of operators is straightforward). In the Schrödinger picture the corrasponding transition amplitude is

$$\text{transition amplitude} = \langle\psi_{f,S}| U_S(t_f,t_1)\,A_{1,S}U_S(t_1,t_2)\,A_{2,S}U_S(t_2,t_i)|\psi_{i,S}\rangle$$

In the Heisenberg and interaction pictures one obtains[9]

$$\text{transition amplitude} = \langle\psi_{f,H}| A_{1,H}(t_1)\,A_{2,H}(t_2)|\psi_{i,H}\rangle$$

while in the interaction picture[10]

$$\text{transition amplitude} = \langle\psi_{f,I}| U(t_f,t_1)\,A_{1,I}(t_1)\,U(t_1,t_2)\,A_{2,I}(t_1)\,U(t_2,t_i)|\psi_{i,I}\rangle$$

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[7] $\langle\psi_f|U_S(t_f,t_i)|\psi_i\rangle = \langle\psi_f|e^{-iHt_f}e^{iHt_i}|\psi_i\rangle = \langle\psi_{f,H}|\psi_{i,H}\rangle$

[8] $\langle\psi_f|U_S(t_f,t_i)|\psi_i\rangle = \langle\psi_{f,I}|e^{iH_0t_f}e^{-iH(t_f-t_i)}e^{-iH_0t_i}|\psi_{i,I}\rangle$ what can be written as $\langle\psi_{f,I}|U(t_f,0)U^{-1}(t_i,0)|\psi_{i,I}\rangle$ followed by $U(t_f,0)U^{-1}(t_i,0) = U(t_f,0)U(0,t_i) = U(t_f,t_i)$

[9] $\langle\psi_f|e^{-iHt_f}e^{iHt_1}A_{1,S}e^{-iHt_1}e^{iHt_2}A_{2,S}e^{-iHt_2}\ldots = \langle\psi_{f,H}|A_{1,H}(t_1)\,A_{2,H}(t_2)\ldots$

[10] $\langle\psi_f|e^{-iHt_f}e^{iHt_1}A_{1,S}\ldots = \langle\psi_{f,I}|e^{iH_0t_f}e^{-iHt_f}e^{iHt_1}e^{-iH_0t_1}A_{1,I}(t_1)e^{iH_0t_1}\ldots = \langle\psi_{f,I}|U(t_f,0)U^{-1}(t_1,0)A_{1,I}(t_1)e^{iH_0t_1}\ldots = \langle\psi_{f,I}|U(t_f,t_1)A_{1,I}(t_1)\ldots$

in and out states

There is a caveat hidden in the transition amplitude written in the Heisenberg picture. The point is that for a given initial state in the Schrödinger picture $|\psi_S(t_i)\rangle = |\psi_i\rangle$, the corresponding $|\psi_{i,H}\rangle$ is usually not known explicitly (and the same holds also for $|\psi_{f,H}\rangle$). Let us consider, e.g. $|\psi_f\rangle = |p_1, p_2\rangle$ and $|\psi_i\rangle = |p_3, p_4\rangle$. Then one may be tempted to write $|\psi_{i,H}\rangle = |p_3, p_4\rangle$, but this would be very misleading. The vector $|p_3, p_4\rangle$ in the Heisenberg picture describes the state which in the Schrödinger picture fulfils $|\psi_S(0)\rangle = |p_3, p_4\rangle$ rather than $|\psi_S(t_i)\rangle = |p_3, p_4\rangle$. So to avoid a notational mismatch, let us emphasize that

$$\begin{aligned}
\text{transition amplitude} &= \langle p_1, p_2 | \, U_S\left(t_f, t_i\right) | p_3, p_4 \rangle \neq \langle p_1, p_2 |_H \;\; | p_3, p_4 \rangle_H \\
\text{transition amplitude} &= \langle p_1, p_2 | \, U_S\left(t_f, t_1\right) A_{1,S} U_S\left(t_1, t_2\right) A_{2,S} U_S\left(t_2, t_i\right) | p_3, p_4 \rangle \\
&\neq \langle p_1, p_2 |_H \;\; A_{1,H}\left(t_1\right) A_{2,H}\left(t_2\right) | p_3, p_4 \rangle_H
\end{aligned}$$

In order to have a simple way to rewrite a transition amplitude from the Schrödinger picture to the Heisenberg one, so-called *in* and *out* states are introduced. They are both defined in the Schrödinger picture as states at $t = 0$, which at the time t_i or t_f are equal to $|\psi_i\rangle$ and $|\psi_f\rangle$ respectively

$$\begin{aligned}
|\psi_{i,\text{in}}\rangle &= |\psi_S(0)\rangle \qquad \text{where} \qquad |\psi_S(t_i)\rangle = |\psi_i\rangle \\
|\psi_{f,\text{out}}\rangle &= |\psi_S(0)\rangle \qquad \text{where} \qquad |\psi_S(t_f)\rangle = |\psi_f\rangle
\end{aligned}$$

It is obvious (using the Schrödinger picture) that

$$\begin{aligned}
|\psi_{i,\text{in}}\rangle &= e^{-iH(0-t_i)} |\psi_i\rangle = e^{iHt_i} |\psi_i\rangle \\
|\psi_{f,\text{out}}\rangle &= e^{-iH(0-t_f)} |\psi_f\rangle = e^{iHt_f} |\psi_f\rangle
\end{aligned}$$

and since these states are defined as states in the Schrödinger picture at $t = 0$, they are equal to the corresponding states in both Heisenberg and interaction pictures[11] $|\psi_{\text{in,out}}\rangle_S = |\psi_{\text{in,out}}\rangle_H = |\psi_{\text{in,out}}\rangle_I$. As to the above example, the correct formulae in the Hesisenberg and interaction pictures are

$$\begin{aligned}
\text{transition amplitude} &= \langle p_1, p_2 |_{\text{out}} \;\; | p_3, p_4 \rangle_{\text{in}} \\
&= \langle p_1, p_2 |_{\text{out}} \;\; U\left(t_f, t_i\right) | p_3, p_4 \rangle_{\text{in}} \\
\text{transition amplitude} &= \langle p_1, p_2 |_{\text{out}} \;\; A_{1,H}\left(t_1\right) A_{2,H}\left(t_2\right) | p_3, p_4 \rangle_{\text{in}} \\
&= \langle p_1, p_2 |_{\text{out}} \;\; U\left(t_f, t_1\right) A_{1,I} U\left(t_1, t_2\right) A_{2,I} U\left(t_2, t_i\right) | p_3, p_4 \rangle_{\text{in}}
\end{aligned}$$

Why to bother with the sophisticated notation in the Heisenberg and interaction pictures, if it anyway refers to the Schrödinger picture? The reason is, of course, that in the relativistic QFT it is preferable to use a covariant formalism, in which field operators depend on time and space-position on the same footing. It is simply preferable to deal with operators $\varphi(x)$ rather than $\varphi(\vec{x})$, which makes the Heisenberg picture more appropriate for relativistic QFT. The Schrödinger picture is most convenient for intuitive grasp of transition amlitudes, the Heisenberg picture is most convenient for formulation of relativistivc field theory and the interaction picture is most convenient for calculations.

[11] Note that the times t_i and t_f refer only to the Schrödinger picture states. Indeed, in spite of what the notation may suggest, the Heisenberg picture *in*- and *out*-states do not change in time. The *in*- and *out*- prefixes have nothing to do with the evolution of states in this picture (there is no such thing in the Heisenberg picture), they are simply labelling conventions (which have everything to do with the time evolution of the corresponding states in the Schrödinger picture).

green functions

For multiparticle systems, a particularly useful set of initial and final states is given by the states of particles simultaneously created at various positions, i.e. by the localized states. But as we have seen already, in the relativistic QFT the more appropriate states are the quasilocalized ones created by the field operators. The corresponding amplitude is apparently something like $\langle 0|\, \varphi_H(\vec{x}_1, t_f)\varphi_H(\vec{x}_2, t_f) \dots \varphi_H(\vec{x}_n, t_i)\, |0\rangle$. The vacuum state in this amplitude, however, is not exactly what it should be.

The time-independent state $|0\rangle$ in the Heisenberg picture corresponds to a particular time-evolving state $|\psi_0(t)\rangle$ in the Schrödinger picture, namely to the one for which $|\psi_0(0)\rangle = |0\rangle$. This state contains no particles at the time $t = 0$. But the fields $\varphi_H(\vec{x}, t_i)$ should rather act on a different state, namely the one which contains no particles at the time t_i. Such a state is nothing else than the previouslz defined $|0\rangle_{\text{in}}$

$$|0\rangle_{\text{in}} = e^{iHt_i}\, |0\rangle$$

In a complete analogy one has to replace the bra-vector $\langle 0|$ by

$$\langle 0|_{\text{out}} = \langle 0|\, e^{-iHt_f}$$

The quantity of interest is therefore given by the product of fields sandwiched not between $\langle 0|$ and $|0\rangle$, but rather

$$\langle 0|_{\text{out}}\, \varphi_H(\vec{x}_1, t_f)\varphi_H(\vec{x}_2, t_f) \dots \varphi_H(\vec{x}_n, t_i)\, |0\rangle_{\text{in}}$$

Because of relativity of simultaneity, however, this quantity looks differently for other observers, namely the time coordinates x_i^0 are not obliged to coincide. These time coordinates, on the other hand, are not completely arbitrary. To any observer the times corresponding to the simultaneous final state in one particular frame, must be all greater than the times corresponding to the simultaneous initial state in this frame. The more appropriate quantity would be a slightly more general one, namely the time-ordered T-product of fields[12] sandwiched between $\langle 0|_{\text{out}}$ and $|0\rangle_{\text{in}}$

$$\langle 0|_{\text{out}}\, T\{\varphi_H(x_1)\varphi_H(x_2) \dots \varphi_H(x_n)\}\, |0\rangle_{\text{in}}$$

The dependence on t_i and t_f is still present in $|0\rangle_{\text{in}}$ and $\langle 0|_{\text{out}}$. It is a common habit to get rid of this dependence by taking $t_i = -T$ and $t_f = T$ with $T \to \infty$

$$g(x_1, \dots, x_n) = \lim_{T\to\infty} \langle 0|\, e^{-iHT} T\left\{\varphi_H(x_1) \dots \varphi_H(x_n)\right\} e^{-iHT}\, |0\rangle$$

The exact reason for this rather arbitrary step remains unclear until the more serious treatment of the whole machinery becomes available in the next section).

The above matrix element is almost, but not quite, the Green function — one of the most prominent quantities in QFT. We shall call these functions the green functions (this notion is not common in literature, but this applies for the whole naive approach presented here). The genuine Green functions G are to be discussed later within the standard approach to the interacting fields (we will distinguish between analogous quantities in the naive and the standard approaches by using lowercase letters in the former and uppercase letter in the latter case).

Actual calculations of the green functions are performed, not surprisingly, in the interaction picture. The transition from the Heisenberg picture to the interaction one is provided by the

[12]For fields commuting at space-like intervals ($[\varphi_H(x), \varphi_H(y)] = 0$ for $(x-y)^2 < 0$) the time ordering is immaterial for times which coincide in a particular reference frame. For time-like intervals, on the other hand, the T-product gives the same ordering in all reference frames.

relations from the page 74. It is useful to start with

$$|0\rangle_{\text{in}} = e^{-iHT}|0\rangle = e^{-iHT}e^{iH_0T}|0\rangle = U^{-1}(-T,0)|0\rangle = U(0,-T)|0\rangle$$
$$\langle 0|_{\text{out}} = \langle 0|e^{-iHT} = \langle 0|e^{iH_0T}e^{-iHT} = \langle 0|U(T,0)$$

which holds for $H_0|0\rangle = 0$. The next step is to use $\varphi_H(x) = U^{-1}(x^0,0)\varphi_I(x)U(x^0,0)$ for every field in the green function, then to write $U(T,0)\varphi_H(x_1)\varphi_H(x_2)\ldots$ as

$$\underbrace{U(T,0)U^{-1}(x_1^0,0)}_{U(T,0)U(0,x_1^0)}\varphi_I(x_1)\underbrace{U(x_1^0,0)U^{-1}(x_2^0,0)}_{U(x_1^0,0)U(0,x_2^0)}\varphi_I(x_2)U(x_2^0,0)\ldots$$

and finally one utilizes $U(T,0)U(0,x_1^0) = U(T,x_1^0)$, etc. This means that for $x_1^0 \geq \ldots \geq x_n^0$ the green function $g(x_1,\ldots,x_n)$ is equal to

$$\lim_{T\to\infty}\langle 0|U(T,x_1^0)\varphi_I(x_1)U(x_1^0,x_2^0)\varphi_I(x_2)\ldots\varphi_I(x_n)U(x_n^0,-T)|0\rangle$$

and analogously for other orderings of times.

Let us now define a slightly generalized time ordered product as

$$T\{U(t,t')A(t_1)B(t_2)\ldots C(t_n)\} = U(t,t_1)A(t_1)U(t_1,t_2)B(t_2)\ldots C(t_n)U(t_n,t')$$

for $t \geq t_1 \geq t_2 \geq \ldots \geq t_n \geq t'$ and for other time orderings the order of operators is changed appropriately. With this definition we can finally write

$$g(x_1,\ldots,x_n) = \lim_{T\to\infty}\langle 0|T\{U(T,-T)\varphi_I(x_1)\ldots\varphi_I(x_n)\}|0\rangle$$

This form of the green function is what we were after. It has the form of the vacuum expectation value of the products of the field operators in the interaction picture and it is relatively straightforward to develop the technique for calculation of these objects. This technique will lead us directly to the Feynman rules. The rules were introduced in the introductory chapter, but they were not derived there. Now we are going to really derive them.

Remark: *The Feynman rules discussed in the Introductions/Conclusions concerned the scattering amplitude M_{fi}, while here we are dealing with the green functions. This, however, represents no contradiction. The green functions, as well as the genuine Green functions, are auxiliary quantities which are, as we will see briefly, closely related to the scattering amplitudes. It is therefore quite reasonable first to formulate the Feynman diagrams for the green or Green functions and only afterwards for the scattering amplitudes.*

perturbation theory

The practically useful, even if only approximate, solution for $U(t,t')$ is obtained by rewriting the differential equation to the integral one

$$U(t,t') = 1 - i\int_{t'}^{t} dt'' H_I(t'')\, U(t'',t')$$

which is then solved iteratively

0^{th} iteration $\quad U(t,t') = 1$

1^{st} iteration $\quad U(t,t') = 1 - i\int_{t'}^{t} dt_1 H_I(t_1)$

2^{nd} iteration $\quad U(t,t') = 1 - i\int_{t'}^{t} dt_1 H_I(t_1) - i^2\int_{t'}^{t} dt_1 H_I(t_1)\int_{t'}^{t_1} dt_2 H_I(t_2)$

etc.

Using a little artificial trick, the whole scheme can be written in a more compact form. The trick is to simplify the integration region in the multiple integrals $I_n = \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \ldots \int_{t'}^{t_{n-1}} dt_n\, H_I(t_1) H_I(t_2) \ldots H_I(t_n)$. Let us consider the n-dimensional hypercube $t' \le t_i \le t$ and for every permutation of the variables t_i take a region for which the first variable varies from t' up to t, the second variable varies from t' up to the first one, etc. There are $n!$ such regions ($n!$ permutations) and any point of the hypercube lies either inside exactly one of these regions, or at a common border of several regions. Indeed, for any point the ordering of the coordinates $(t_1,\ldots,t_n)$, from the highest to the lowest one, reveals unambiguously the region (or a border) within which it lies. The integral of the product of Hamiltonians over the whole hypercube is equal to the sum of integrals over the considered regions. In every region one integrates the same product of Hamiltonians, but with different ordering of the times. The trick now is to force the time ordering to be the same in all regions. This is achieved in a rather artificial way, namely by introducing the so-called time-ordered product $T\{A(t_1)\,B(t_2)\ldots\}$, which is the product with the terms organized from the left to the right with respect to decreasing time (the latest on the very left etc.). Integrals of this T-product over different regions are equal to each other, so we can replace the original integrals by the integrals over hypercubes $I_n = \frac{1}{n!}\int_{t'}^{t}\int_{t'}^{t} dt_1 \ldots dt_n\; T\{H_I(t_1)\ldots H_I(t_n)\}$ and consequently

$$U(t,t') = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} \ldots \int_{t'}^{t} dt_1 \ldots dt_n\, T\{H_I(t_1)\ldots H_I(t_n)\}$$

which is usually written in a compact form as

$$U(t,t') = Te^{-i\int_{t'}^{t} dt''\, H_I(t'')}$$

where the definition of the RHS is the RHS of the previous equation.

Note that if the interaction Hamiltonian is proportional to some constant (e.g. a coupling constant) then this iterative solution represents the power expansion (perturbation series[13]) in this constant.

[13]The usual time-dependent perturbation theory is obtained by inserting the expansion $|\psi_I(t)\rangle = a_n(t)\,|\varphi_n\rangle$, where $|\varphi_n\rangle$ are eigenvectors of the free Hamiltonian H_0, into the original equation $i\partial_t|\psi_I\rangle = H_I(t)\,|\psi_I\rangle$. From here one finds a differential equation for $a_n(t)$, rewrites it as an integral equation and solves it iteratively.

Wick's theorem

The perturbative expansion of $U(T,-T)$ is a series in $H_I(t)$, which in turn is a functional of $\varphi_I(x)$, so our final expression for the green function gives them as a series of VEVs (vacuum expectation values) of products of φ_I-fields.

As we already know from the Introductions, the most convenient way of calculating the VEVs of products of creation and annihilation operators is to rush the creation and annihilation operators to the left and to the right respectively. We are now going to accommodate this technique to the VEVs of time-ordered products of φ_I-fields.

The keyword is *the normal product* of fields. First one writes $\varphi_I = \varphi_I^+ + \varphi_I^-$, where φ_I^+ and φ_I^- are parts of the standard expansion of $\varphi_I(x)$ containing only the annihilation and the creation operators respectively[14]. The normal product of fields, denoted as $N\{\varphi_I(x)\varphi_I(y)\ldots\}$ or $:\varphi_I(x)\varphi_I(y)\ldots:$, is defined as the product in which all φ_I^--fields are reshuffled by hand to the left of all φ_I^+-fields, e.g. $N\{\varphi_I(x)\varphi_I(y)\} = \varphi_I^-(x)\varphi_I^-(y) + \varphi_I^-(x)\varphi_I^+(y) + \varphi_I^-(y)\varphi_I^+(x) + \varphi_I^+(x)\varphi_I^+(y)$. Everybody likes normal products, because their VEVs vanish.

The trick, i.e. the celebrated Wick's theorem, concerns the relation between the time-ordered and normal products. For two fields one has $\varphi_I(x)\varphi_I(y) = N\{\varphi_I(x)\varphi_I(y)\} + \left[\varphi_I^+(x), \varphi_I^-(y)\right]$. It is straightforward to show (do it) that $\left[\varphi_I^+(x), \varphi_I^-(y)\right] = D(x-y)$ where

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}e^{-ip(x-y)}$$

The relation between $T\{\varphi_I\varphi_I\}$ and $N\{\varphi_I\varphi_I\}$ is now straightforward

$$T\{\varphi_I(x)\varphi_I(y)\} = N\{\varphi_I(x)\varphi_I(y)\} + d_F(x-y)$$

where

$$d_F(\xi) = \vartheta\left(\xi^0\right)D(\xi) + \vartheta\left(-\xi^0\right)D(-\xi)$$

The function d_F is almost equal to the so-called Feynman propagator D_F (see p.86). Everybody likes $d_F(x-y)$, $D_F(x-y)$ and similar functions, because they are not operators and can be withdrawn out of VEVs.

For three fields one obtains in a similar way[15]

$$\begin{aligned}\varphi_I(x)\varphi_I(y)\varphi_I(z) &= N\{\varphi_I(x)\varphi_I(y)\varphi_I(z)\}\\ &\quad + D(x-y)\varphi_I(z) + D(x-z)\varphi_I(y) + D(y-z)\varphi_I(x)\\ T\{\varphi_I(x)\varphi_I(y)\varphi_I(z)\} &= N\{\varphi_I(x)\varphi_I(y)\varphi_I(z)\}\\ &\quad + d_F(x-y)\varphi_I(z) + d_F(x-z)\varphi_I(y) + d_F(y-z)\varphi_I(x)\end{aligned}$$

Now we can formulate and prove the Wick's theorem for n fields

$$\begin{aligned}T\{\varphi_I(x_1)\ldots\varphi_I(x_n)\} &= N\{\varphi_I(x_1)\ldots\varphi_I(x_n)\}\\ &\quad + d_F(x_1-x_2)\,N\{\varphi_I(x_3)\ldots\varphi_I(x_n)\} + \ldots\\ &\quad + d_F(x_1-x_2)\,d_F(x_3-x_4)\,N\{\varphi_I(x_5)\ldots\varphi_I(x_n)\} + \ldots\\ &\quad + \ldots\end{aligned}$$

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[14] Strange, but indeed $\varphi_I^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}a_{\vec{p}}e^{-ipx}$ and $\varphi_I^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}a_{\vec{p}}^+e^{ipx}$. The superscript $\pm$ is not in honour of the creation and annihilation operators, but rather in honour of the sign of energy $E = \pm\omega_{\vec{p}}$.

[15] One starts with $\varphi_I(x)\varphi_I(y)\varphi_I(z) = \varphi_I(x)\varphi_I(y)\varphi_I^+(z) + \varphi_I(x)\varphi_I(y)\varphi_I^-(z)$, followed by $\varphi_I(x)\varphi_I(y)\varphi_I^-(z) = \varphi_I(x)[\varphi_I(y),\varphi_I^-(z)] + [\varphi_I(x),\varphi_I^-(z)]\varphi_I(y) + \varphi_I^-(z)\varphi_I(x)\varphi_I(y)$, so that $\varphi_I(x)\varphi_I(y)\varphi_I(z) = \varphi_I(x)\varphi_I(y)\varphi_I^+(z) + \varphi_I(x)D(y-z) + D(x-z)\varphi_I(y) + \varphi_I^-(z)\varphi_I(x)\varphi_I$. At this point one utilizes the previous result for two fields, and finally one has to realize that $N\{\varphi_I(x)\varphi_I(y)\}\varphi_I^+(z) + \varphi_I^-(z)N\{\varphi_I(x)\varphi_I(y)\} = N\{\varphi_I(x)\varphi_I(y)\varphi_I(z)\}$.

In each line the ellipsis stands for terms equivalent to the first term, but with the variables x_i permutated in all possible ways. The number of the Feynman propagators is increased by one when passing to the next line. The proof is done by induction, the method is the same as a we have used for three fields.

The most important thing is that except for the very last line, the RHS of the Wick's theorem has vanishing VEV (because of normal products). For n odd even the last line has vanishing VEV, for n even the VEV of the last line is an explicitly known number. This gives us the quintessence of the Wick's theorem: for n odd $\langle 0|\, T\left\{\varphi_I\left(x_1\right)\ldots\varphi_I\left(x_n\right)\right\}|0\rangle = 0$, while for n even

$$\langle 0|\, T\left\{\varphi_I\left(x_1\right)\ldots\varphi_I\left(x_n\right)\right\}|0\rangle = d_F\left(x_1-x_2\right)\ldots d_F\left(x_{n-1}-x_n\right) + \text{permutations}$$

Remark: *It is a common habit to economize a notation in the following way. Instead of writing down the products $d_F\left(x_1-x_2\right)\ldots d_F\left(x_{n-1}-x_n\right)$ one writes the product of fields and connects by a clip the fields giving the particular d_F. In this notation*

$$d_F\left(x_1-x_2\right) d_F\left(x_3-x_4\right) = \overbrace{\varphi_I\left(x_1\right)\varphi_I\left(x_2\right)}\,\overbrace{\varphi_I\left(x_3\right)\varphi_I\left(x_4\right)}$$
$$d_F\left(x_1-x_3\right) d_F\left(x_2-x_4\right) = \overbrace{\varphi_I\left(x_1\right)\underbrace{\varphi_I\left(x_2\right)\varphi_I\left(x_3\right)}\varphi_I\left(x_4\right)}$$
$$d_F\left(x_1-x_4\right) d_F\left(x_2-x_3\right) = \overbrace{\varphi_I\left(x_1\right)\underbrace{\varphi_I\left(x_2\right)\varphi_I\left(x_3\right)}\varphi_I\left(x_4\right)}$$

At this point we are practically done. We have expressed the green functions as a particular series of VEVs of time-ordered products of φ_I-operators and we have learned how to calculate any such VEV by means of the Wick's theorem. All one has to do now is to expand $U(T,-T)$ in the green function up-to a given order and then to calculate the corresponding VEVs.

Example: *$g\left(x_1,x_2,x_3,x_4\right)$ in the φ^4-theory*
notation: $\varphi_i := \varphi_I\left(x_i\right)$, $\varphi_x := \varphi_I\left(x\right)$, $d_{ij} := d_F(x_i-x_j)$, $d_{ix} := d_F(x_i-x)$

$$g = \lim_{T\to\infty}\langle 0|\, T\left\{U(T,-T)\varphi_1\varphi_2\varphi_3\varphi_4\right\}|0\rangle = g^{(0)} + g^{(1)} + \ldots$$
$$g^{(0)} = \langle 0|\, T\left\{\varphi_1\varphi_2\varphi_3\varphi_4\right\}|0\rangle = d_{12}d_{34} + d_{13}d_{24} + d_{14}d_{23}$$
$$g^{(1)} = -\frac{ig}{4!}\,\langle 0|\, T\left\{\int d^4x\,\varphi_x^4\varphi_1\varphi_2\varphi_3\varphi_4\right\}|0\rangle$$
$$= -\frac{ig}{4!}\int d^4x\left\{24\times d_{1x}d_{2x}d_{3x}d_{4x} + 12\times d_{12}d_{xx}d_{3x}d_{4x} + \ldots\right\}$$

where we have used $U(\infty,-\infty) = 1 - i\int_{-\infty}^{\infty} dt\, H_I\left(t\right) + \ldots = 1 - \frac{ig}{4!}\int d^4x\,\varphi_x^4 + \ldots$

Feynman rules

The previous example was perhaps convincing enough in two respects: first that in principle the calculations are quite easy (apart from integrations, which may turn out to be difficult), and second that practically they become almost unmanageable rather soon (in spite of our effort to simplify the notation). Conclusion: further notational simplifications and tricks are called for urgently.

The most wide-spread trick uses a graphical representation of various terms in green function expansion. Each variable x is represented by a point labeled by x. So in the previous example we would have 4 points labeled by x_i $i = 1, 2, 3, 4$ and furthermore a new point $x, x', \ldots$ for each power of H_I. Note that the Hamiltonian density $\mathcal{H}_I$ contains several fields, but all at the same point — this is the characteristic feature of local theories. For each $d_F(y-z)$ the points labeled by y and z are connected by a line. If there are several different fields, there are several different Feynman propagators, and one has to use several different types of lines.

In this way one assigns a diagram to every term supplied by the team-work of the perturbative expansion of $U(T,-T)$ and the Wick's theorem. Such diagrams are nothing else but the famous Feynman diagrams. Their structure is evident from the construction. Every diagram has external points, given by the considered g-function, and internal points (vertices), given by $\mathcal{H}_I$. The number of internal points is given by the order in the perturbative expansion. The structure of the vertices (the number and types of lines entering the vertex) is given by the structure of H_I, each product of fields represent a vertex, each field in the product represent a line entering this vertex.

A diagram, by construction, represents a number. This number is a product of factors corresponding to lines and vertices. The factor corresponding to a line (internal or external) connecting x, y is $d_F(x-y)$. The factor corresponding to a vertex in the above example is $-\frac{ig}{4!}\int d^4x$, while in the full generality it is

$$-i \times \text{what remains of } \mathcal{H}_I \text{ after the fields are "stripped off"} \times \int d^4x$$

Further simplification concerns combinatorics. Our procedure, as described so-far, gives a separate diagram for each of the 24 terms $-\frac{ig}{4!}\int d^4x\, d_{1x}d_{2x}d_{3x}d_{4x}$ in $g^{(1)}$ in the above example. As should be clear from the example, this factor is purely combinatorial and since it is typical rather than exceptional, it is reasonable to include this 24 into the vertex factor (and to draw one diagram instead of 24 identical diagrams). This is achieved by doing the appropriate combinatorics already in the process of "stripping the fields off", and it amounts to nothing more than to the multiplication by $n!$ for any field appearing in $\mathcal{H}_I$ in the n-th power. An economic way of formalizing this "stripping off" procedure, with the appropriate combinatorics factors, is to use the derivatives of $\mathcal{H}_I$ with respect to the fields.

Having included the typical combinatorial factor into the vertex, we have to pay a special attention to those (exceptional) diagrams which do not get this factor. The 12 terms $-\frac{ig}{4!}\int d^4x\, d_{12}d_{xx}d_{3x}d_{4x}$ in $g^{(1)}$ in the example can serve as an illustration. Twelve identical diagrams are represented by one diagram according to our new viewpoint, but this diagram is multiplied by 24, hidden in the vertex factor, rather then by 12. To correct this, we have to divide by 2 — one example of the infamous explicit combinatorial factors of Feynman rules.
The rules can be summarized briefly as

the Feynman rules for the green functions in the x-representation

line (internal or external)	$d_F(x-y)$
vertex (n legs)	$-i\,\frac{\partial^n \mathcal{H}_I}{\partial \varphi_I^n}\Big\vert_{\varphi_I=0}\int d^4x$

These are not the Feynman rules from the Introductions yet, but we are on the right track.

The first step towards the rules from the Introductions concerns the relation between $\mathcal{H}_I$ and $\mathcal{L}_{\text{int}}$. For interaction Lagrangians with no derivative terms (like the φ^4-theory), the definition $H = \int d^3x\ (\dot{\varphi}\pi - \mathcal{L})$ implies immediately $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$. And since $\mathcal{H}_{\text{int}}$ in the Heisenberg

picture is the same function of φ_H-fields, as $\mathcal{H}_I$ is of φ_I-fields (as we have convinced ourselves), one can replace $-\partial^n \mathcal{H}_I/\partial\varphi_I^n$ by $\partial^n \mathcal{L}_{\text{int}}/\partial\varphi^n$. Finally, for vertices one can replace $\mathcal{L}_{\text{int}}$ by $\mathcal{L}$ in the last expression, because the difference is the quadratic part of the Lagrangian, and vertices under consideration contain at least three legs. For interactions with derivative terms (say $\mathcal{L}_{\text{int}} \sim \varphi\partial_\mu\varphi\partial^\mu\varphi$) the reasoning is more complicated, but the result is the same. We will come back to this issue shortly (see p.83). For now let us proceed, as directly as possible, with the easier case.

Another step is the use of the Fourier expansion[16]

$$d_F(x-y) = \int \frac{d^4p}{(2\pi)^4} d_F(p)\, e^{-ip(x-y)}$$

This enables us to perform the vertex x-integrations explicitly, using the identity $\int d^4x\, e^{-ix(p+p'+\ldots)} = (2\pi)^4 \delta^4(p+p'+\ldots)$, what results in

the Feynman rules for the green functions in the p-representation

internal line	$\int \frac{d^4p}{(2\pi)^4} d_F(p)$
external line	$\int \frac{d^4p}{(2\pi)^4} d_F(p)\, e^{\pm ipx_i}$
vertex	$i \left.\frac{\partial^n \mathcal{L}}{\partial\varphi^n}\right\vert_{\varphi=0} (2\pi)^4 \delta^4(p+p'+\ldots)$

Let us remark that some authors prefer to make this table simpler-looking, by omitting the factors of $(2\pi)^4$ as well as the momentum integrations, and shifting them to the additional rule requiring an extra $(2\pi)^4$ for each vertex and $(2\pi)^{-4} \int d^4p$ for each line (internal or external). We have adopted such a convention in the Introductions.

derivative couplings

Now to the interaction Lagrangians with derivative terms. The prescription from the Introductions was quite simple: any ∂_μ in the interaction Lagrangian furnishes the $-ip^\mu$ factor for the corresponding vertex in the p-representation Feynman rules (p^μ being the momentum assigned to the corresponding leg, oriented toward the vertex). To understand the origin of this factor, it is (seemingly) sufficient to differentiate the Wick's theorem, e.g. for two fields

$$T\left\{\varphi_I(x)\,\partial'_\mu\varphi_I(x')\right\} = \partial'_\mu\left(N\left\{\varphi_I(x)\,\varphi_I(x')\right\} + d_F(x-x')\right)$$

When calculating the green function, the derivative can be withdrawn from VEV, and once $d_F(x-x')$ is Fourier expanded, it produces the desired factor (the reader is encouraged to make him/her-self clear about momentum orientations).

[16] One may be tempted to use $i(p^2-m^2)^{-1}$ or $i(p^2-m^2+i\varepsilon)^{-1}$ as $d_F(p)$ (see p. 86), but neither would be correct. Both choices lead to results differing from $d_F(x-y)$ by some functions of ε, which tend to disappear when $\varepsilon \to 0$. Nevertheless the ε-differences are the important ones, they determine even the seemingly ε-independent part of the result.

Anyway, apart from the $i\varepsilon$ subtleties, $d_F(p)$ comes out equal to what we have calculated in the Introductions (from quite different definition of propagator). This may seem like a coincidence, and one may suspect if one gets equal results even beyond the real scalar field example. The answer is affirmative, but we are not going to prove it here in the full generality. The reason is that the general statement is more transparent in another formulation of QFT, namely in the path integral formalism. So we prefer to discuss this issue within this formalism.

There is, however, a subtlety involved. The above identity is not straightforward, even if it follows from the straightforward identity $\varphi_I(x)\partial'_\mu\varphi_I(x') = \partial'_\mu(\varphi_I(x)\varphi_I(x')) = \partial'_\mu(N\varphi_I(x)\varphi_I(x') + D(x-x'))$. The point is that when combining two such identities to get the T-product at the LHS, one obtains $\vartheta(\xi^0)\partial'_\mu D(\xi) + \vartheta(-\xi^0)\partial'_\mu D(-\xi)$ instead of $\partial_\mu d_F(\xi)$ on the RHS (with $\xi = x - x'$). The extra term, i.e. the difference between what is desired and what is obtained, is $D(\xi)\partial_0\vartheta(\xi^0) + D(-\xi)\partial_0\vartheta(-\xi^0) = (D(\xi) - D(-\xi))\delta(\xi^0)$ and this indeed vanishes, as can be shown easily from the explicit form of $D(\xi)$ (see page 80).

Unfortunately, this is not the whole story. Some extra terms (in the above sense) are simply die-hard. They do not vanish as such, and one gets rid of them only via sophisticated cancellations with yet another extras entering the game in case of derivative couplings[17]. Attempting not to oppress the reader, we aim to outline the problem, without penetrating deeply into it.

The troublemaker is the T-product of several differentiated fields. An illustrative example is provided already by two fields, where one obtains

$$T\left\{\partial_\mu\varphi_I(x)\,\partial'_\nu\varphi_I(x')\right\} = \partial_\mu\partial'_\nu\left(N\left\{\varphi_I(x)\,\varphi_I(x')\right\} + d_F(x-x')\right) + \delta^0_\mu\delta^0_\nu\Delta(x-x')$$

with[18] $\Delta(\xi) = -i\delta^4(\xi)$. The same happens in products of more fields and the Wick's theorem is to be modified by the non-vanishing extra term $-i\delta^0_\mu\delta^0_\nu\delta^4(\xi)$, on top of the doubly differentiated standard propagator. In the "clip notation"

$$\partial_\mu\overbrace{\varphi_I(x)\,\partial'_\nu\varphi_I}(x') = \partial_\mu\partial'_\nu\overbrace{\varphi_I(x)\,\varphi_I}^{\text{standard}}(x') + \overbrace{\varphi_I(x)\,\varphi_I}^{\text{extra}}(x')$$

where $\overbrace{\varphi_I(x)\,\varphi_I}^{\text{standard}}(x') = d_F(x-x')$ and $\overbrace{\varphi_I(x)\,\varphi_I}^{\text{extra}}(x') = -i\delta^0_\mu\delta^0_\nu\delta^4(x-x')$

The rather unpleasant feature of this extra term is its non-covariance, which seems to ruin the highly appreciated relativistic covariance of the perturbation theory as developed so-far.

Because of the δ-function, the extra term in the propagator can be traded for an extra vertex. To illustrate this, let us consider as an example $\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{g}{2}\varphi\partial_\mu\varphi\partial^\mu\varphi$. A typical term in the perturbative expansion of a green function contains $\mathcal{H}_{\text{int}}[\varphi(x)]\,\mathcal{H}_{\text{int}}[\varphi(x')]$ and clipping the fields together via the extra term gives[19]

$$\frac{-ig}{2}\varphi(x)\,\partial^\mu\varphi(x)\,\overbrace{\varphi_I(x)\,\varphi_I}^{\text{extra}}(x')\frac{-ig}{2}\varphi(x')\,\partial^\nu\varphi(x') = i\frac{g^2}{4}\varphi^2(x)\,\dot\varphi^2(x)$$

effectively contracting two original vertices into the new extra one. In this way one can get rid of the extra non-covariant term in the propagator, at the price of introduction of the non-covariant effective vertex. In our example this effective vertex corresponds to an extra term in the Lagrangian: $\mathcal{L}_{\text{extra}} = \frac{1}{2}g^2\varphi^2\dot\varphi^2$.

The factor $\frac{1}{2}$ follows from a bit of combinatorics. There are four possibilities for the extra clipping between the two $\mathcal{H}_{\text{int}}$, endowing the new effective vertex with the factor of 4. Less obvious is another factor of $\frac{1}{2}$, coming from the fact that interchange of the two contracted

[17] Similar problems (and similar solutions) haunt also theories of quantum fields with higher spins, i.e. they are not entirely related to derivative couplings.

[18] First one gets, along the same lines as above, $\Delta(\xi) = (D(\xi)-D(-\xi))\partial_0\delta(\xi^0)+2\delta(\xi^0)\partial_0(D(\xi)-D(-\xi))$. Due to the identity $f(x)\delta'(x) = -f'(x)\delta(x)$ this can be brought to the form $\Delta(\xi) = \delta(\xi^0)\partial_0(D(\xi)-D(-\xi))$ and plugging in the explicit form of $D(\xi)$ one obtains $\Delta(\xi) = \delta(\xi^0)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec p}}\partial_0(e^{-ip\xi}-e^{ip\xi}) = \delta(\xi^0)\int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{2\omega_{\vec p}}(e^{-ip\xi}+e^{ip\xi}) = -i\delta(\xi^0)\int\frac{d^3p}{(2\pi)^3}e^{i\vec p\vec\xi} = -i\delta^4(\xi)$.

[19] Here we pretend that $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$, which is not the whole truth in the case at hand. We will correct this in the moment.

original vertices does not change the diagram. According to the rules for combinatoric factors (see section**??**) this requires the factor of $\frac{1}{2}$. Once the vertices are contracted, there is no (combinatoric) way to reconstruct this factor, so it has to be included explicitly.

The story is not over yet. There is another source of non-covariant vertices. The point is that once derivative couplings are present, the canonical momentum is not equal to the time derivative of the field any more. As an illustration let us consider our example again. Here one gets $\pi = \dot{\varphi} + g\varphi\dot{\varphi}$, i.e. $\dot{\varphi} = (1+g\varphi)^{-1}\pi$. The corresponding Hamiltonian density can be written as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$ where[20]

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2$$
$$\mathcal{H}_{\text{int}} = \frac{g}{2}\varphi|\nabla\varphi|^2 - \frac{g}{2}\varphi(1+g\varphi)^{-1}\pi^2$$

$\mathcal{H}_0$ corresponds to the Hamiltonian density of the free field, expressed in terms of conjugate quantities, obeying (after quantization) the standard commutation relation $[\varphi(x), \pi(y)] = i\delta^3(\vec{x}-\vec{y})$. Using this $\mathcal{H}_0$ one can develop the perturbation theory in the standard way. Doing so it is convenient, as we have seen, to re-express the canonical momentum in terms of the field variables, leading to

$$\mathcal{H}_{\text{int}} = -\frac{1}{2}g\varphi\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}g^2\varphi^2\dot{\varphi}^2$$

As announced, this interaction Hamiltonian density contains, on top of the expected covariant term $-\mathcal{L}_{\text{int}}$, a non-covariant one. But now, the fanfares breaks out, and the non-covariant vertices originating from two different sources, cancel each other.[21] This miracle is not an exceptional feature of the example at hand, it is rather a general virtue of the canonical quantization: at the end of the day all non-covariant terms in vertices and propagators tend to disappear.

[20] $\mathcal{H} = \dot{\varphi}\pi - \mathcal{L} = \dot{\varphi}\pi - \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2 - \frac{g}{2}\varphi\partial_\mu\varphi\partial^\mu\varphi$
$= \frac{1}{2}(1+g\varphi)^{-1}\pi^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2 + \frac{g}{2}\varphi|\nabla\varphi|^2$
$= \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2 + \frac{g}{2}\varphi|\nabla\varphi|^2 - \frac{g}{2}\varphi(1+g\varphi)^{-1}\pi^2$

[21] One may worry about what happens to the non-covariant part of $\mathcal{H}_{\text{int}}$ contracted (with whatever) via the non-covariant part of the propagator. Indeed, we have not consider such contractions, but as should be clear from what was said so-far, for any such contraction there is a twin contraction with opposite sign, so all such terms cancels out.

propagator

In the Introductions/Conclusions we have learned that the propagator of the scalar field is equal to $i/(p^2-m^2)$. Let us check now, whether this ansatz for $d_F(p)$ really leads to the correct expression for $d_F(\xi)$, i.e. if

$$d_F(\xi) \stackrel{?}{=} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip\xi}$$

where $d_F(\xi) = \vartheta\left(\xi^0\right) D(\xi) + \vartheta\left(-\xi^0\right) D(-\xi)$ and $D(\xi) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip\xi}$.

It is very useful to treat the p_0-variable in this integral as a complex variable. Writing $p^2-m^2 = (p_0-\omega_{\vec{p}})(p_0+\omega_{\vec{p}})$ (recall that $\omega_{\vec{p}} = \sqrt{\vec{p}^2+m}$) one finds that the integrand has two simple poles in the p^0-variable, namely at $p^0 = \pm\omega_{\vec{p}}$ with the residua $\pm(2\omega_{\vec{p}})^{-1} e^{\mp i\omega_{\vec{p}}\xi_0} e^{i\vec{p}.\vec{\xi}}$. The integrand is, on the other hand, sufficiently small at the lower (upper) semicircle in the p^0-plane for $\xi^0 > 0$ ($\xi^0 < 0$), so that it does not contribute to the integral for the radius of the semicircle going to infinity. So it almost looks like if

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip\xi} \stackrel{?}{=} \pm \int \frac{d^3p}{(2\pi)^3} \left(\frac{e^{-ip\xi}}{p^0+\omega_{\vec{p}}}|_{p^0=\omega_{\vec{p}}} + \frac{e^{-ip\xi}}{p^0-\omega_{\vec{p}}}|_{p^0=-\omega_{\vec{p}}} \right)$$

(the sign reflects the orientation of the contour) which would almost give the desired result after one inserts appropriate ϑ-functions and uses $\vec{p} \to -\vec{p}$ substitution in the last term.

Now, was that not for the fact that the poles lay on the real axis, one could perhaps erase the questionmarks safely. But since they do lay there, one can rather erase the equality sign.

It is quite interesting, however, that one can do much better if one shifts the poles off the real axis. Let us consider slightly modified ansatz for the propagator, namely $i/(p^2-m^2+i\varepsilon)$ with positive ε (see the footnote on the page **??**). The pole in the variable p_0^2 lies at $\omega_{\vec{p}}^2 - i\varepsilon$, i.e. the poles in the variable p_0 lie at $\omega_{\vec{p}} - i\varepsilon$ and $-\omega_{\vec{p}} + i\varepsilon$ and so trick with the complex plane now works perfectly well, leading to (convince yourself that it really does)

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\xi} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left\{ \vartheta(\xi^0) e^{-ip\xi-\varepsilon\xi_0} + \vartheta(-\xi^0) e^{ip\xi+\varepsilon\xi_0} \right\}$$

At this point one may be tempted to send ε to zero and then to claim the proof of the identity $d_F(p) = i/(p^2-m^2)$ being finished. This, however, would be very misleading. The limit $\varepsilon \to 0^+$ is quite non-trivial and one cannot simply replace ε by zero (that is why we were not able to take the integral in the case of $\varepsilon = 0$).

Within the naive approach one cannot move any further. Nevertheless, the result is perhaps sufficient to suspect the close relation between the result for the propagator as found in the Introductions/Conclusions and in the present chapter. Later on we will see that the $i\varepsilon$ prescription is precisely what is needed when passing from the naive approach to the standard one.

s-matrix

The green functions, discussed so-far, describe the time evolution between the initial and final states of particles created at certain positions. Most experimental setups in the relativistic particle physics correspond to different settings, namely to the initial and final states of particles created with certain momenta in the remote past and the remote future respectively. We will therefore investigate now the so-called s-matrix (almost, but not quite the famous S-matrix)

$$s_{fi} = \lim_{T\to\infty} \langle \vec{p}_1, \ldots, \vec{p}_m | \, U(T,-T) \, | \vec{p}_{m+1}, \ldots, \vec{p}_n \rangle$$

where f and i are abbreviations for $\vec{p}_1, \ldots, \vec{p}_m$ and $\vec{p}_{m+1}, \ldots, \vec{p}_n$ respectively. We have presented the definition directly in the interaction picture, which is most suitable for calculations. Of course, it can be rewritten in any other picture, as discussed on p.**??**.

The difference between the s-matrix and the genuine S-matrix (which is to be discussed within the standard approach) is in the states between which $U(T,-T)$ is sandwiched. Within our naive approach we adopt a natural and straightforward choice, based on the relation[22] $|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}} a^+_{\vec{p}} \, |0\rangle$, leading to

$$s_{fi} = \lim_{T\to\infty} \langle 0 | \sqrt{2\omega_{\vec{p}_1}} a_{\vec{p}_1,I}(T) \ldots U(T,-T) \ldots \sqrt{2\omega_{\vec{p}_n}} a^+_{\vec{p}_n,I}(-T) \, |0\rangle$$

Intuitively this looks quite acceptable, almost inevitable: the multi-particle state is created, by the corresponding creation operators, from the state with no particles at all. There is, however a loophole in this reasoning.

The main motivation for the s-matrix was how do the real experiments look like. The states entering the definition should therefore correspond to some typical states prepared by accelerators and detected by detectors. The first objection which may come to one's mind is that perhaps we should not use the states with sharp momenta (plane waves) but rather states with "well-defined, even if not sharp" momenta and positions (wave-packets). This, however, is not the problem. One can readily switch from plane-waves to wave-packets and vice versa, so the difference between them is mainly the difference in the language used, rather than a matter of principle.

The much more serious objection is this one: Let us suppose that we have at our disposal apparatuses for measurement of momenta. Then we can prepare states more or less close to the states with sharp energy and 3-momentum. The above considered states $|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}} a^+_{\vec{p}} \, |0\rangle$ are such states, but only for the theory with the free Hamiltonian. Once the interaction is switched on, the said $|\vec{p}\rangle$ states may differ significantly from what is prepared by the available experimental devices. Once more and aloud: typical experimentally accessible states in the worlds with and without interaction may differ considerably. And, as a rule, they really do.

One may, of course, ignore this difference completely. And it is precisely this ignorance, what constitutes the essence of our naive approach. Indeed, the core of this approach is the work with the s-matrix. defined in terms of explicitly known simple states, instead of dealing with the S-matrix, defined in terms of the states experimentally accessible in the real world (with interactions). The latter are usually not explicitly known, so the naivity simplifies life a lot.

What excuse do we have for such a simplification? Well, if the interaction may be viewed as only a small perturbation of the free theory — and we have adopted this assumption already,

[22] The relation is given in a bit sloppy way. In the Schrödinger picture, it is to be understood as $|\vec{p}, -T\rangle_S = \sqrt{2\omega_{\vec{p}}} a^+_{\vec{p},S} \, |0\rangle$, where $|0\rangle$ is just a particular state in the Fock space (no time dependence of $|0\rangle$ is involved in this relation). In the interaction picture the relation reads $|\vec{p}, -T\rangle_I = \sqrt{2\omega_{\vec{p}}} a^+_{\vec{p},I}(-T) \, |0\rangle$ (this is equivalent to the Schrödinger picture due to the fact that $H_0 \, |0\rangle = 0$). In the Heisenberg picture, however, one has $|\vec{p}\rangle_H = \sqrt{2\omega_{\vec{p}}} a^+_{\vec{p},H}(-T) \, e^{-iHT} \, |0\rangle \neq \sqrt{2\omega_{\vec{p}}} a^+_{\vec{p},H}(-T) \, |0\rangle$ (due to the fact that $|0\rangle$ is usually not an eigenstate of H).

namely in the perturbative treatment of $U(T,-T)$ operator — then one may hope that the difference between the two sets of states is negligible. To take this hope too seriously would be indeed naive. To ignore it completely would be a bit unwise. If nothing else, the s-matrix is the zeroth order approximation to the S-matrix, since the unperturbed states are the zeroth order approximation of the corresponding states in the full theory. Moreover, the developments based upon the naive assumption tend to be very useful, one can get pretty far using this assumption and almost everything will survive the more rigorous treatment of the standard approach.

As to the calculation of the s-matrix elements, it follows the calculation of the green functions very closely. One just uses the perturbative expansion $U(T,-T) = T\exp\{-i\int_{-T'}^{T} dt\ H_I(t)\}$ (see p.79) and the Wick's theorem, which is to be supplemented by[23]

$$\varphi_I(x)\, a^+_{\vec{p},I}(-T) = N\{\varphi_I(x)\, a^+_{\vec{p},I}(-T)\} + \frac{1}{\sqrt{2\omega_{\vec{p}}}} e^{-ipx} e^{-i\omega_{\vec{p}}T}$$
$$a_{\vec{p},I}(T)\, \varphi_I(x) = N\{a_{\vec{p},I}(T)\, \varphi_I(x)\} + \frac{1}{\sqrt{2\omega_{\vec{p}}}} e^{ipx} e^{-i\omega_{\vec{p}}T}$$

or in the "clipping notation"

$$\overbrace{\varphi_I(x)\, a_{\vec{p},I}}^{+}(-T) = \frac{1}{\sqrt{2\omega_{\vec{p}}}} e^{-ipx} e^{-i\omega_{\vec{p}}T}$$
$$\overbrace{a_{\vec{p},I}(T)\, \varphi_I}(x) = \frac{1}{\sqrt{2\omega_{\vec{p}}}} e^{ipx} e^{-i\omega_{\vec{p}}T}$$

Consequently, the s-matrix elements are obtained in almost the same way as are the green functions, i.e. by means of the Feynman rules. The only difference is the treatment of the external lines: instead of the factor $d_F(x-y)$, which was present in the case of the green functions, the external legs provide the factors $e^{\mp ipx} e^{-i\omega_{\vec{p}}T}$ for the s-matrix elements, where the upper and lower sign in the exponent corresponds to the ingoing and outgoing particle respectively. (Note that the $\sqrt{2\omega_{\vec{p}}}$ in the denominator is canceled by the $\sqrt{2\omega_{\vec{p}}}$ in the definition of s_{fi}.) In the Feynman rules the factor $e^{-i\omega_{\vec{p}}T}$ is usually omitted, since it leads to the pure phase factor $\exp\{-iT\sum_{i=1}^{n}\omega_{\vec{p}_n}\}$, which is redundant for probability densities, which we are interested in[24].

[23] Indeed, first of all one has $\varphi_I(x)\, a^+_{\vec{p},I}(t') = N\{\varphi_I(x) a^+_{\vec{p},I}(t')\} + [\varphi_I(x), a^+_{\vec{p},I}(t')]$ and then $[\varphi_I(x), a^+_{\vec{p},I}(t')] = \int \frac{d^3p'}{(2\pi)^3} \frac{e^{i\vec{p}'\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}'}}} [a_{\vec{p}',I}(t), a^+_{\vec{p},I}(t')] = \int \frac{d^3p'}{\sqrt{2\omega_{\vec{p}'}}} e^{i(\vec{p}'\cdot\vec{x} - \omega_{\vec{p}'}t + \omega_{\vec{p}}t')} \delta(\vec{p}-\vec{p}')$ and the same gymnastics (with an additional substitution $p' \to -p'$ in the integral) is performed for $[a_{\vec{p},I}(t'), \varphi_I(x)]$.

[24] Omission of the phase factor is truly welcome, otherwise we should bother about the ill-defined limit $T \to \infty$. Of course, avoiding problems by omitting the trouble-making pieces is at least nasty, but what we are doing here is not that bad. Our sin is just a sloppiness. We should consider, from the very beginning, the limit $T \to \infty$ for the probability and not for the amplitude.

In this way one obtains

the Feynman rules for the s-matrix elements in the x-representation

internal line	$d_F(x-y)$
ingoing external line	$e^{\mp ipx}$
vertex (n legs)	$-i\left.\frac{\delta^n \mathcal{L}}{\delta \varphi^n}\right\vert_{\varphi=0} \int d^4x$

The next step is the use of the Fourier expansion of $d_F(x-y)$ allowing for explicit x-integrations (see p.83), resulting in

the Feynman rules for the s-matrix elements in the p-representation

internal line	$\int \frac{d^4p}{(2\pi)^4}\, d_F(p)$
external line	1
vertex	$i\left.\frac{\delta^n \mathcal{L}}{\delta \varphi^n}\right\vert_{\varphi=0} (2\pi)^4\, \delta^4(p+p'+\ldots)$

Omitting the factors of $(2\pi)^4$ as well as the momentum integrations, and shifting them to the additional rule requiring an extra $(2\pi)^4$ for each vertex and $(2\pi)^{-4} \int d^4p$ for each internal line, one obtains

another form of the Feynman rules
for the s-matrix elements in the p-representation

internal line	$d_F(p)$
external line	1
vertex	$i\left.\frac{\delta^n \mathcal{L}}{\delta \varphi^n}\right\vert_{\varphi=0} \delta^4(p+p'+\ldots)$

which is now really very close to our presentation of the Feynman rules in the Introductions/Conclusions[25].

[25] In the Introductions/Conclusions the Feynman rules were used to calculate the scattering amplitude M_{fi} rather than the S-matrix. These two quantities are, however, closely related: $S_{fi} = \mathbf{1} + iM_{fi}(2\pi)^4 \delta^{(4)}(P_f - P_i)$ or $s_{fi} = \mathbf{1} + im_{fi}(2\pi)^4 \delta^{(4)}(P_f - P_i)$.

connected diagrams

IIThe M_{fi}, or rather m_{fi} within our naive approach, is more appropriate for the discussion of the cross-sections and decay rates, which is our next task.At this point, there are only three differences left:

- presence of $d_F(p)$ instead of the genuine Feynman propagator $D_F(p)$
- no $\sqrt{Z}$ factors corresponding to external legs
- presence of disconnected diagrams like

in the perturbative expansions of the green function and the s-matrix[26], while in the Introductions/Conclusions only connected Feynman diagrams were accounted for.

The differences are due to the fact that we are dealing with the s-matrix rather than the S-matrix. In the next section we will learn how the so-far missed ingredients (replacement of d_F by D_F, appearance of $\sqrt{Z}$ and fadeaway of disconnected diagrams) will enter the game in the standard approach.

As to the comparison of the rules presented in the Introductions/Conclusions to the ones derived here, let us remark that in the Introductions/Conclusions we did not introduce the notion of the S-matrix explicitly. Neverthweless, it was present implicitly via the quantity M_{fi}, since S and M are very closely related

$$S_{fi} = \mathbf{1} + iM_{fi}(2\pi)^4 \delta^{(4)}(P_f - P_i)$$

The M_{fi}, or rather m_{fi} within our naive approach, is more appropriate for the discussion of the cross-sections and decay rates, which is our next task.

Remark: *As we have seen, the green functions g and the s-matrix elements s_{fi} are very closely related. The only differences are the external legs factors: $d_F(p)$ for the green function g and simply 1 (or something slightly more complicated in case of higher spins) for the s-matrix elements. This may be formulated in the following way: s_{fi} is obtained from the corresponding green function g by multiplication of each external leg by the inverse propagator. Another, even more popular, formulation: s_{fi} is obtained from the corresponding g by amputation of the external legs. Actually, the relation between s and g is of virtually no interest whatsoever. We are, after all, interested only in the s-matrix, so there is no reason to bother about the green functions. Indeed, we could ignore the whole notion of the green function and derive the rules directly for the s-matrix. Doing so, however, we would miss the nice analogy between the naive and the standard approaches.*
The point is that similar relation holds also for the genuine Green functions G and the S-matrix elements. Indeed, as we will see, S_{fi} is obtained from the corresponding Green function G by amputation of the external leg and multiplication by $\sqrt{Z}$. And in the standard approach, unlike in the naive one, one cannot easily avoid the Green functions when aiming at the S-matrix.

[26] Which term in the perturbative expansion of $U(T,-T)$ corresponds this diagram to?

remark on complex fields and arrows

The expansion of the complex scalar field in creation and annihilation operators reads

$$\varphi_I(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}(t)\,e^{i\vec{p}.\vec{x}} + b^+_{\vec{p}}(t)\,e^{-i\vec{p}.\vec{x}}\right)$$
$$\varphi^*_I(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a^+_{\vec{p}}(t)\,e^{-i\vec{p}.\vec{x}} + b_{\vec{p}}(t)\,e^{i\vec{p}.\vec{x}}\right)$$

with

$$\left[a_{\vec{p}}(t), a^+_{\vec{p}'}(t)\right] = (2\pi)^3\,\delta(\vec{p}-\vec{p}')$$
$$\left[b_{\vec{p}}(t), b^+_{\vec{p}'}(t)\right] = (2\pi)^3\,\delta(\vec{p}-\vec{p}')$$

and all other commutators of creation and annihilation operators equal to zero. It is now straightforward to show that

$$T\{\varphi_I(x)\,\varphi_I(y)\} = N\{\varphi_I(x)\,\varphi_I(y)\}$$
$$T\{\varphi^*_I(x)\,\varphi^*_I(y)\} = N\{\varphi^*_I(x)\,\varphi^*_I(y)\}$$
$$T\{\varphi_I(x)\,\varphi^*_I(y)\} = N\{\varphi_I(x)\,\varphi^*_I(y)\} + d_F(x-y)$$
$$T\{\varphi^*_I(x)\,\varphi_I(y)\} = N\{\varphi^*_I(x)\,\varphi_I(y)\} + d_F(x-y)$$

This means that the time ordered product of two φ_I-fields as well as of the two φ^*_I-fields is already in the normal form, i.e. the only contributions to the Feynman diagrams come from $T\{\varphi_I\varphi^*_I\}$ and $T\{\varphi^*_I\varphi_I\}$.

This result is typical for complex fields: they provide two types of propagators, corresponding to products of the field and the conjugate field in two possible orderings. In case of the complex scalar field the factors corresponding to the two different orderings are equal to each other, so there is no reason to use two different graphical representations. This, however, is not a general feature. In other cases (e.g. in case of the electron-positron field) the different orderings lead to different factors. It is therefore necessary to distinguish these two possibilities also in their graphical representation, and this is usually done by means of an arrow.

3.1.3 Cross-sections and decay rates

Because of the relativistic normalization of states $\langle\vec{p}|\vec{p}'\rangle = 2E_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{p}')$ the S-matrix (s-matrix)[27] elements do not give directly the probability amplitudes. To take care of this, one has to use the properly normalized vectors $(2\pi)^{-3/2}(2E)^{-1/2}|\vec{p}\rangle$, which leads to the probability amplitude equal to S_{fi} multiplied by $\prod_{j=1}^{n}(2\pi)^{-3/2}(2E_j)^{-1/2}$. Taking the module squared one obtains

$$\text{probability density} = |S_{fi}|^2\prod_{j=1}^{n}\frac{1}{(2\pi)^3 2E_j}$$

This expression presents an unexpected problem. The point is that S_{fi} turns out to contain δ-functions, and so $|S_{fi}|^2$ involves the ill-defined square of the δ-function. The δ-functions originate from the normalization of states and they are potentially present in any calculation which involves states normalized to the δ-function. In many cases one is lucky enough not to encounter any such δ-function in the result, but sometimes one is faced with the problem of dealing with δ-functions in probability amplitudes. The problem, when present, is usually treated either by switching to the finite volume normalization, or by exploitation of the specific set of basis vectors ("wave-packets" rather than "plane waves").

There are two typical δ-functions occurring in S_{fi}. The first one is just the normalization δ-function, symbolically written as $\delta(f-i)$, which represents the complete result in the case of the free Hamiltonian (for initial and final states being eigenstates of the free Hamiltonian). In the perturbative calculations, this $\delta(f-i)$ remains always there as the lowest order result. It is therefore a common habit to split the S-matrix as (the factor i is purely formal)

$$S_{fi} = \delta(f-i) + iT_{fi}$$

and to treat the corresponding process in terms of T_{fi}, which contains the complete information on transition probability for any $|f\rangle \neq |i\rangle$. The probability for $|f\rangle = |i\rangle$ can be obtained from the normalization condition (for the probability). Doing so, one effectively avoids the square of $\delta(f-i)$ in calculations.

But even the T-matrix is not "δ-free", it contains the momentum conservation δ-function. Indeed, both the (full) Hamiltonian and the 3-momentum operator commute with the time-evolution operator $0 = [P^\mu, U(T,-T)]$. When sandwiched between some 4-momentum eigenstates[28] $\langle f|$ and $|i\rangle$, this implies $(p_f^\mu - p_i^\mu)\langle f|U(T,-T)|i\rangle = 0$, and consequently $(p_f^\mu - p_i^\mu)S_{fi} = 0$. The same, of course, must hold for T_{fi}. Now any (generalized) function of f and i, vanishing for $p_f^\mu \neq p_i^\mu$, is either a non-singular function (finite value for $p_f = p_i$), or a distribution proportional to $\delta^4(p_f - p_i)$, or even a more singular function (proportional to some derivative of $\delta^4(p_f-p_i)$). We will assume proportionality to the δ-function

$$T_{fi} = (2\pi)^4\delta^4(p_f - p_i)M_{fi}$$

where $(2\pi)^4$ is a commonly used factor. Such an assumption turns out to lead to the finite result. We will ignore the other two possibilities, since if the δ-function provides a final result, they would lead to either zero or infinite results.

Now back to the notational mismatch. When discussing the first δ-function $\delta(f-i)$, the states $\langle f|$ and $|i\rangle$ were eigenstates of the free Hamiltonian H_0. On the other hand, when discussing the second δ-function $\delta^4(p_f-p_i)$, the states $\langle f|$ and $|i\rangle$ were eigenstates of the full Hamiltonian

[27] All definitions of this paragraph are formulated for the S-matrix, but they apply equally well for the s-matrix, e.g. one have $s_{fi} = \delta(f-i) + it_{fi}$ and $t_{fi} = (2\pi)^4\delta^4(p_f-p_i)m_{fi}$.

[28] The notational mismatch, introduced at this very moment, is to be discussed in a while.

H. We should, of course, make just one unambiguous choice of notation, and in this naive approach the choice is: $\langle f|$ and $|i\rangle$ are eigenstates of the free Hamiltonian H_0. This choice invalidates a part of the above reasoning, namely the part leading to $\delta(E_f - E_i)$ (the part leading to $\delta^3(\vec{p}_f - \vec{p}_i)$ remains untouched, since H_0 commutes with $\vec{P}$ and so one can choose $\langle f|$ and $|i\rangle$ to be eigenstates of both H_0 and $\vec{P}$).

Nevertheless, we are going to use $\delta^4(p_f - p_i)$ (rather then $\delta^3(\vec{p}_f - \vec{p}_i)$)in the definition of M_{fi}. The point is that $\delta(E_f - E_i)$ can be present in T_{fi}, even if the above justification fails. That this is indeed the case (to any order of the perturbation theory) can be understood directly from the Feynman rules. Recall that in the p-representation every vertex contains the momentum δ-function. Every propagator, on the other hand, contains the momentum integration and after all these integrations are performed, one is left with just one remaining δ-function, namely $\delta^4(p_f - p_i)$.

Proof: take a diagram, ignore everything except for momentum δ-functions and integrations. Take any two vertices connected directly by an internal line and perform the corresponding integration, using one of the vertices δ-functions. After integration the remaining δ-function contains momenta of all legs of the both vertices. The result can be depicted as a diagram with two vertices shrunk into a new one, and the new vertex contains the appropriate momentum δ-function. If the considered vertices were directly connected by more than one internal line, then the new diagram contains some internal lines going from the new vertex back to itself. From the point of view of the present argument, such "daisy-buck loops" can be shrunk to the point, since they do not contain any δ-function. The procedure is then iterated until one obtains the simplest possible diagram with one vertex with all external lines attached to it and with the corresponding δ-function $\delta(\Sigma\, p_{\text{ext}})$. The final point is to realize that in Feynman diagrams momenta are oriented towards the vertices, while final state momenta are usually understood as flowing from the diagram, i.e. $\Sigma\, p_{\text{ext}} = p_f - p_i$.

After having identified the typical δ-function present in the T-matrix, we should face the annoying issue of the undefined square of this δ-function in $|T_{fi}|^2$. We will do this in a covardly manner, by an attempt to avoid the problem by the standard trick with the universe treated as a finite cube (length L) with the periodic boundary conditions. The allowed 3-momenta are $\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$ and the 3-momentum δ-function transfers to

$$\delta^3(\vec{p} - \vec{p}') = \int \frac{d^3x}{(2\pi)^3} e^{i(\vec{p}-\vec{p}').\vec{x}} \to \frac{V}{(2\pi)^3}\delta_{\vec{p}\vec{p}'} \stackrel{\text{def}}{=} \delta^3_V(\vec{p} - \vec{p}')$$

while the same trick performed with the time variable gives the analogous result $\delta(E - E') \to \frac{T}{2\pi}\delta_{EE'} \stackrel{\text{def}}{=} \delta_T(E - E')$. The calculation of $|T_{fi}|^2$ in the finite 4-dimensional cube presents no problem at all: $|T_{fi}|^2 = V^2T^2\delta_{\vec{p}_f\vec{p}_i}\delta_{E_fE_i}|M_{fi}|^2$. For reasons which become clear soon, we will write this as

$$|T_{fi}|^2 = VT(2\pi)^4\delta^4_{VT}(p_f - p_i)|M_{fi}|^2$$

where $\delta^4_{VT} = \delta^3_V\delta_T$.

The finite volume normalization affects also the normalization of states, since $\langle\vec{p}|\vec{p}'\rangle = 2E_{\vec{p}}(2\pi)^3\delta^3(\vec{p} - \vec{p}') \to 2E_{\vec{p}}V\delta_{\vec{p}\vec{P}'}$. The relation from the beginning of this paragraph between $|S_{fi}|^2$ (or $|T_{fi}|^2$ for $f \neq i$) and the corresponding probability therefore becomes: $\text{probability}_{\text{for } f\neq i} = |T_{fi}|^2\prod_{j=1}^n \frac{1}{2E_jV}$. Note that using the finite volume normalization, i.e. having discrete rather than continuous labeling of states, we should speak about probabilities rather than probability densities. Nevertheless, for the volume V big enough, this discrete distribution of states is very dense — one may call it quasi-continuous. In that case it is technically convenient

to work with the probability quasi-density, defined as the probability of decay into any state $|f\rangle = |\vec{p}_1, \ldots, \vec{p}_m\rangle$ within the phase-space element $d^3p_1 \ldots d^3p_m$. This is, of course, nothing else but the sum of the probabilities over all states within this phase-space element. If all the probabilities within the considered phase-space element were equal (the smaller the phase-space element is, the better is this assumption fulfilled) one could calculate the said sum simply by multiplying this probability by the number of states within the phase-space element. And this is exactly what is commonly used (the underlying reasoning is just a quasi-continuous version of the standard reasoning of integral calculus). And since the number of states within the interval d^3p is $\Delta n_x \Delta n_y \Delta n_z = V\, d^3p/\,(2\pi)^3$ one comes to the probability quasi-density (for $f \neq i$) being equal to $|T_{fi}|^2 \prod_{j=1}^{n} \frac{1}{2E_jV} \prod_{k=1}^{m} \frac{V\, d^3p}{(2\pi)^3}$. So for $f \neq i$ one has

$$\begin{matrix}\text{probability}\\ \text{quasidensity}\end{matrix} = VT\,(2\pi)^4\,\delta^4_{VT}(p_f - p_i)\,|M_{fi}|^2 \prod_{j=1}^{m} \frac{d^3p}{(2\pi)^3\,2E_j} \prod_{j=m+1}^{n} \frac{1}{2E_jV}.$$

Comparing this to the expressions for $d\Gamma$ and $d\sigma$ given in the Introductions (see p.18) we realize that we are getting really close to the final result. We just have to get rid of the awkward factors T and T/V in the probability quasi-densities for one and two initial particles respectively (and to find the relation between $1/E_AE_B$ and $[(p_A.p_B)^2 - m_A^2m_B^2]^{-1/2}$ in case of the cross section). This step, however, is quite non-trivial. The point is that even if our result looks as if we are almost done, actually we are almost lost. Frankly speaking, the result is absurd: for the time T being long enough, the probability exceeds 1.

At this point we should critically reexamine our procedure and understand the source of the unexpected obscure factor T. Instead, we are going to follow the embarrassing tradition of QFT textbooks and use this evidently unreliable result for further reasoning, even if the word reasoning used for what follows is a clear euphemism[29]. The reason for this is quite simple: the present author is boldly unable to give a satisfactory exposition of these issues[30].

After having warned the reader about the unsoundness of what follows, we can proceed directly to the interpretation of the result for one-particle initial state in terms of the decay rate: From the linear time dependence of the probability density one can read out the probability density per unit time, this is equal to the time derivative of the probability density and for the exponential decay $\exp(-\Gamma t)$ this derivative taken at $t = 0$ is nothing else but the decay rate Γ (or $d\Gamma$ if we are interested only in decays with specific final states).

The previous statement is such a dense pack of lies that it would be hardly outmatched in an average election campaign. First of all, we did not get the linear time dependence, since T is not the "flowing time", but rather a single moment. Second, even if we could treat T as a variable, it definitely applies only to large times and, as far as we can see now, has absolutely nothing to say about infinitesimal times in the vicinity of $t = 0$. Third, if we took the linear time dependence seriously, then why to speak about exponential decay. Indeed, there is absolutely no indication of the exponential decay in our result.

[29] A fair comment on rates and cross sections is to be found in the Weinberg's book (p.134): *The proper way to approach these problems is by studying the way that experiments are actually done, using wave packets to represent particles localized far from each other before a collision, and then following the time history of these superpositions of multiparticle states. In what follows we will instead give a quick and easy derivation of the main results, actually more a mnemonic than a derivation, with the excuse that (as far as I know) no interesting open questions in physics hinge on getting the fine points right regarding these matters.*

[30] This incapability seems to be shared by virtually all authors of QFT textbooks (which perhaps brings some relief to any of them). There are many attempts, more or less related to each other, to introduce decay rates and cross sections. Some of them use finite volume, some use wave-packets (but do not closely follow the whole time evolution, as suggested by Weinberg's quotation), some combine the two approaches. And one feature is common to all of them: they leave much to be desired.

Nevertheless, for the reasons unclear at this point (they are discussed in the appendix **??**) the main conclusion of the above lamentable statement remains valid: decay rate is given by the probability quasi-density divided by T. After switching back to the infinite volume, which boils down to $\delta^4_{VT}(p_f - p_i) \to \delta^4(p_f - p_i)$, one obtains

$$d\Gamma = (2\pi)^4 \delta^4 (P_f - P_i) \frac{1}{2E_A} |M_{fi}|^2 \prod_{i=1}^{m} \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

where E_A is the energy of the decaying particle.

Remark: *The exponential decay of unstable systems is a notoriously known matter, perhaps too familiar to realize how non-trivial issue it becomes in the quantum theory. Our primary understanding of the exponential decay is based on the fact that* $\exp(-\Gamma t)$ *is the solution of the simple differential equation* $dN/dt = -\Gamma N(t)$*, describing a population of individuals diminishing independently of a) each other b) the previous history. In quantum mechanics, however, the exponential decay should be an outcome of the completely different time evolution, namely the one described by the Schrödinger equation.*
Is it possible to have the exponential decay in the quantum mechanics, i.e. can one get $|\langle \psi_0 | \psi(t) \rangle|^2 = e^{-\Gamma t}$ *for* $|\psi(t)\rangle$ *being a solution of the Schrödinger equation with the initial condition given by* $|\psi_0\rangle$*? The answer is affirmative, e.g. one can easily convince him/herself that for the initial state* $|\psi_0\rangle = \sum c_\lambda |\varphi_\lambda\rangle$*, where* $H|\varphi_\lambda\rangle = E_\lambda |\varphi_\lambda\rangle$*, one obtains the exponential decay for* $\sum |c_\lambda|^2 \delta(E - E_\lambda) \stackrel{\text{def}}{=} p(E) = \frac{1}{2\pi} \frac{\Gamma}{(E-E_0)^2 + \Gamma^2/4}$ *(the so-called Breit-Wigner distribution of energy in the initial state).*
The Breit-Wigner distribution could nicely explain the exponential decay in quantum mechanics, were it not for the fact that this would require understanding of why this specific distribution is so typical for quantum systems. And this is very far from being obvious. Needless to say, the answer cannot be that the Breit-Wigner is typical because the exponential decay is typical, since this would immediately lead to a tautology. It would be nice to have a good understanding for the exponential decay in the quantum mechanics, but (unfortunately) we are not going to provide any.

For two particles in the initial state the reasoning is a bit more reasonable. The main trick is to use another set of the initial and final states, the one which enables semiclassical viewpoint, which further allows to give some sense to the suspicious factor T/V. We are speaking about well localized wave-packets with well defined momentum (to the extend allowed by the uncertainty principle). Let us call one particle the target, while the second one the beam. The target will be localized in all three space dimensions, the beam is localized just in one direction — the one of their relative momentum $\vec{p}_A - \vec{p}_B$. In the perpendicular directions the particle is completely unlocalized, the state corresponds to the strictly zero transverse momentum, in this way the particle simulates the transversely uniform beam. Let us note that if we want (as we do) to simulate a beam with a constant density independent of the extensiveness of the universe box, the beam particle state is to be normalized to L^2 rather than to 1.

Now in the finite-box-universe with the periodic boundary conditions the beam particle leaves the box from time to time, always simultaneously entering on the other side of the world. During the time T the scattering can therefore take place repeatedly. In the rest frame of the target particle the number of scatterings is $\frac{T}{L/v}$, where v is the velocity of the beam particle in this frame. The reasonable quantity (the cross-section[31]) in the target rest frame is therefore the probability quasi-density as obtained above, divided by the "repetition factor" vT/L and multiplied by the beam normalization factor L^2

$$d\sigma = (2\pi)^4 \, \delta^4_{VT}(p_f - p_i) \frac{1}{4E_A E_B v} \left|M_{fi}\right|^2 \prod_{j=1}^{m} \frac{d^3p}{(2\pi)^3 \, 2E_j}$$

Finally, one switches back to the infinite universe by $\delta^4_{VT}(p_f - p_i) \to \delta^4(p_f - p_i)$.

The energies E_A and E_B, as well as the beam particle velocity v, is understood in the target rest frame. To have a formula applicable in any frame, one should preferably find a Lorentz scalar, which in the target rest frame becomes equal to $E_A E_B v$. Such a scalar is provided by $[(p_A.p_B)^2 - m_A^2 m_B^2]^{1/2}$, since for $p_A = (m_A, \vec{0})$ it equals to $m_A(E_B^2 - m_B^2)^{1/2} = m_A \left|\vec{p}_B\right|$ and $v = \left|\vec{p}_B\right| / E_B$, and so we have come to the final result

$$d\sigma = (2\pi)^4 \, \delta^4 \left(P_f - P_i\right) \frac{\left|M_{fi}\right|^2}{4\sqrt{(p_A.p_B)^2 - m_A^2 m_B^2}} \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 \, 2E_i}$$

[31] Reacall that the cross-section is a (semi)classical notion, defined as follows. For a uniform beam of (semi)classical particles interacting with a uniformly distributed target particles, the number dn of beam particles scattered into an element of phase space dP_f is proportional to the flux j of the beam (density×velocity), the number N of the particles in target and dP_f itself: $dn \propto jNdP_f$. The coefficient of proportionality is the cross-section.

3.2 Standard approach

3.2.1 Free particles

the bare propagator equal to the dressed one?

double role of the free Hamiltonian

problem in transition amplitudes - states of one Hamiltonian, operator in terms of the other Hamiltonian

the $i\varepsilon$-trick

$$e^{-iHT}\left|\psi\right\rangle = \sum_n e^{-iE_nT}\left|\varphi_n\right\rangle\left\langle\varphi_n\right|\left.\psi\right\rangle$$

$$\left|\varphi_0\right\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iHT}\left|\psi\right\rangle}{e^{-iE_0T}\left\langle\varphi_0\right|\left.\psi\right\rangle}$$

Instead of taking the bra vector corresponding to this ket one, it will be more convenient to take

$$\left\langle\psi\right| e^{-iHT} = \sum_n \left\langle\psi\right.\left|\varphi_n\right\rangle\left\langle\varphi_n\right| e^{-iE_nT}$$

$$\left\langle\varphi_0\right| = \lim_{T\to\infty(1-i\varepsilon)} \frac{\left\langle\psi\right| e^{-iHT}}{e^{-iE_0T}\left\langle\psi\right.\left|\varphi_0\right\rangle}$$

we will need the product of denominators

$$\lim_{T\to\infty(1-i\varepsilon)} e^{-iE_02T}\left\langle\psi\right|\left.\varphi_0\right\rangle\left\langle\varphi_0\right|\left.\psi\right\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left\langle\psi\right| e^{-iH2T}\sum_n \left|\varphi_n\right\rangle\left\langle\varphi_n\right|\left.\psi\right\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left\langle\psi\right| e^{-iH2T}\left|\psi\right\rangle$$

If $\left\langle\varphi_0\right|\left.\psi\right\rangle$ then $\left|\varphi_1\right\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iHT}\left|\psi\right\rangle}{e^{-iE_1T}\left\langle\varphi_1\right|\left.\psi\right\rangle}$, etc.

Green functions

$$G\left(x_1,\ldots,x_n\right) = \lim_{T\to\infty} \left\langle\Omega\right| T\left\{\varphi_H(x_1)\ldots\varphi_H(x_n)\right\}\left|\Omega\right\rangle$$

$$G\left(x_1,\ldots,x_n\right) = \lim_{T\to\infty(1-i\varepsilon)} \frac{\left\langle 0\right| e^{-iHT} T\left\{\varphi_H(x_1)\ldots\varphi_H(x_n)\right\} e^{-iHT}\left|0\right\rangle}{\left\langle 0\right| e^{-iH2T}\left|0\right\rangle}$$

how is the trick implemented? propagator

the S-matrix

wave packets

$$S_{fi} = \lim_{T\to\infty} \left\langle\vec{p}_1,\ldots,\vec{p}_m\right| U\left(T,-T\right)\left|\vec{p}_{m+1},\ldots,\vec{p}_n\right\rangle$$

$$\left|\vec{p}\right\rangle = \sqrt{2E_{\vec{p}}^{\text{eff}}}\, a_{\vec{p}}^{+\text{eff}}\left|\Omega\right\rangle$$

$$\left|\vec{p}\right\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iHT} a_{\vec{p}}^{+}\left|0\right\rangle}{e^{-iE_{\vec{p}}^{\text{eff}}T}\left\langle\vec{p}\right| a_{\vec{p}}^{+}\left|0\right\rangle}$$

The denominator is just the square root of the dresed propagator.

Part II

Quantum Electrodynamics (Photons and Electrons)

Part III

The Standard Model (Leptons and Quarks)

Appendices

It is not our intention, however, to abandon the non-relativistic case completely. We will return to it, at least briefly, once the Feynman diagrams machinery is fully developed in the relativistic case. Then we will be able to grasp the non-relativistic case rather quickly, just because of strong similarities[32].

[32]This is definitely not the best way for anybody interested primarily in the non-relativistic many-body quantum theory. For such a reader it would be preferable to develop the diagram technique already here. Such a development would include a couple of fine points which, when understood well in the non-relativistic case, could be subsequently passed through quickly in the relativistic case. So it would make a perfect sense not to stop here.

Anyway, it seems to be an unnecessary luxury to discuss everything in detail in both versions (relativistic and non-relativistic). The point is that the details are subtle, difficult and quite a few. Perhaps only one version should be treated in all details, the other one can be then glossed over. And in these lectures, the focus is on the relativistic version.

Appendix A

The bird's-eye view of the solid state physics

Even if we are not going to entertain ourselves with *calculations* within the creation and annihilation operators formalism of the non-relativistic many-body QM, we may still want to spend some time on *formulation* of a specific example. The reason is that the example may turn out to be quite instructive. This, however, is not guaranteed. Moreover, the content of this section is not requisite for the rest of the text. The reader is therefore encouraged to skip the section in the first (and perhaps also in any subsequent) reading.

The basic approximations of the solid state physics

A solid state physics deals with a macroscopic number of nuclei and electrons, interacting electromagnetically. The dominant interaction is the Coulomb one, responsible for the vast majority of solid state phenomena covering almost everything except of magnetic properties. For one type of nucleus with the proton number Z (generalization is straightforward) the Hamiltonian in the wave-function formalism[1] is $H = -\sum_i \frac{\Delta_i}{2M} - \sum_j \frac{\Delta_j}{2m} + U(\vec{R}) + V(\vec{r}) + W(\vec{R},\vec{r})$, where $U(\vec{R}) = \frac{1}{8\pi}\sum_{i\neq j} \frac{Z^2e^2}{|\vec{R}_i-\vec{R}_j|}$, $V(\vec{r}) = \frac{1}{8\pi}\sum_{i\neq j} \frac{e^2}{|\vec{r}_i-\vec{r}_j|}$, $W(\vec{R},\vec{r}) = -\frac{1}{8\pi}\sum_{i,j} \frac{Ze^2}{|\vec{R}_i-\vec{r}_j|}$ and (obviously) $\vec{R} = (\vec{R}_1,\ldots)$, $\vec{r} = (\vec{r}_1,\ldots)$. If desired, more than Coulomb can be accounted for by changing U, V and W appropriately.

Although looking quite innocent, this Hamiltonian is by far too difficult to calculate physical properties of solids directly from it. Actually, no one has ever succeeded in proving even the basic experimental fact, namely that nuclei in solids build a lattice. Therefore, any attempt to grasp the solid state physics theoretically, starts from the series of clever approximations.

The first one is the Born-Oppenheimer adiabatic approximation which enables us to treat electrons and nuclei separately. The second one is the Hartree-Fock approximation, which enables us to reduce the unmanageable many-electron problem to a manageable single-electron problem. Finally, the third of the celebrated approximations is the harmonic approximation, which enables the explicit (approximate) solution of the nuclei problem.

These approximations are used in many areas of quantum physics, let us mention the QM of molecules as a prominent example. There is, however, an important difference between the use

[1] If required, it is quite straightforward (even if not that much rewarding) to rewrite the Hamiltonian in terms of the creation and annihilation operators (denoted as uppercase and lowercase for nuclei and electrons respectively) $H = \int d^3p \, \left(\frac{p^2}{2M}\, A^+_{\vec{p}} A_{\vec{p}} + \frac{p^2}{2m}\, a^+_{\vec{p}} a_{\vec{p}}\right) + \frac{e^2}{8\pi}\int d^3x\, d^3y \frac{1}{|\vec{x}-\vec{y}|}\, \left(Z^2 A^+_{\vec{x}} A^+_{\vec{y}} A_{\vec{y}} A_{\vec{x}} - Z A^+_{\vec{x}} a^+_{\vec{y}} a_{\vec{y}} A_{\vec{x}} + a^+_{\vec{x}} a^+_{\vec{y}} a_{\vec{y}} a_{\vec{x}}\right)$.

of the approximations for molecules and for solids. The difference is in the number of particles involved (few nuclei and something like ten times more electrons in molecules, Avogadro number in solids). So while in the QM of molecules, one indeed solves the equations resulting from the individual approximations, in case of solids one (usually) does not. Here the approximations are used mainly to setup the conceptual framework for both theoretical analysis and experimental data interpretation.

In neither of the three approximations the Fock space formalism is of any great help (although for the harmonic approximation, the outcome is often formulated in this formalism). But when moving beyond the three approximations, this formalism is by far the most natural and convenient one.

Born-Oppenheimer approximation treats nuclei and electrons on different footing. The intuitive argument is this: nuclei are much heavier, therefore they would typically move much slower. So one can perhaps gain a good insight by solving first the electron problem for nuclei at fixed positions (these positions, collectively denoted as $\vec{R}$, are understood as parameters of the electron problem)

$$H_e(\vec{R})\psi_n(\vec{r}) = \varepsilon_n(\vec{R})\psi_n(\vec{r})$$
$$H_e(\vec{R}) = -\sum_i \frac{\Delta_i}{2m} + V + W$$

and only afterwards to solve the nucleus problem with the electron energy $\varepsilon_n(\vec{R})$ understood as a part of the potential energy on the nuclei

$$H_N\Psi_m(\vec{R}) = \mathcal{E}_m\Psi_m(\vec{R})$$
$$H_N = -\sum_i \frac{\Delta_i}{2M} + U + \varepsilon_n(\vec{R})$$

The natural question as to which electron energy level (which n) is to be used in H_N is answered in a natural way: one uses some kind of mean value, typically over the canonical ensemble, i.e. the $\varepsilon_n(\vec{R})$ in the definition of the H_N is to be replaced by $\bar{\varepsilon}(\vec{R}) = \sum_n \varepsilon_n(\vec{R}) \exp\{-\varepsilon_n(\vec{R})/kT\}$.

The formal derivation of the above equations is not important for us, but we can nevertheless sketch it briefly. The eigenfunctions $\Phi_m(\vec{R},\vec{r})$ of the full Hamiltonian H are expanded in terms of the complete system (in the variable $\vec{r}$) of functions $\psi_n(\vec{r})$: $\Phi_m(\vec{R},\vec{r}) = c_{mn}(\vec{R})\psi_n(\vec{r})$. The coefficients of this expansion are, of course, $\vec{R}$-dependent. Plugging into $H\Phi_m = E_m\Phi_m$ one obtains the equation for $c_{mn}(\vec{R})$ which is, apart of an extra term, identical to the above equation for $\Psi_m(\vec{R})$. The equation for $c_{mn}(\vec{R})$ is solved iteratively, the zeroth iteration ignores the extra term completely. As to the weighted average $\bar{\varepsilon}(\vec{R})$ the derivation is a bit more complicated, since it has to involve the statistical physics from the beginning, but the idea remains unchanged.

The Born-Oppenheimer adiabatic approximation is nothing else but the zeroth iteration of this systematic procedure. Once the zeroth iteration is solved, one can evaluate the neglected term and use this value in the first iteration. But usually one contents oneself by checking if the value is small enough, which should indicate that already the zeroth iteration was good enough. In the solid state physics one usually does not go beyond the zeroth approximation, not even check whether the neglected term comes out small enough, simply because this is too difficult.

In the QM of molecules, the Born-Oppenheimer approximation stands behind our qualitative understanding of the chemical binding, which is undoubtedly one of the greatest and the most far-reaching achievements of QM. Needless to say, this qualitative understanding is supported by

impressive quantitative successes of quantum chemistry, which are based on the same underlying approximation. In the solid state physics, the role of this approximation is more modest. It serves only as a formal tool allowing for the separation of the electron and nucleus problems. Nevertheless, it is very important since it sets the basic framework and paves the way for the other two celebrated approximations.

Hartree-Fock approximation replaces the many-electron problem by the related single-electron problem. The intuitive argument is this: even if there is no guarantee that one can describe the system reasonably in terms of states of individual electrons, an attempt is definitely worth a try. This means to restrict oneself to electron wave-functions of the form $\psi(\vec{r}) = \prod_i \varphi_i(\vec{r}_i)$. The effective Hamiltonian for an individual electron should look something like

$$H_{\mathrm{Hartree}} = -\frac{\Delta_i}{2m} + W(\vec{R}, \vec{r}_i) + \sum_{j \neq i} \frac{e^2}{8\pi} \int d^3 r_j \frac{\varphi_j^*(\vec{r}_j)\varphi_j(\vec{r}_j)}{|\vec{r}_i - \vec{r}_j|}$$

where the last term stands for the potential energy of the electron in the electrostatic field of the rest of the electrons.

The Schrödinger-like equation $H\varphi_i(\vec{r}_i) = \varepsilon_i \varphi_i(\vec{r}_i)$ is a non-linear equation which is solved iteratively. For a system of n electrons one has to find the first n eigenvalues ε, corresponding to mutually orthogonal eigenfunctions φ_i, so that the multi-electron wave-function ψ does not contain any two electrons in the same state (otherwise it would violate the Pauli exclusion principle). The resulting ψ, i.e. the limit of the iteration procedure, is called the self-consistent Hartree wave-function.

In the Hartree approximation, however, the Pauli principle is not accounted for sufficiently. The multi-electron wave-function is just the product of single-electron wave-functions, not the anti-symmetrized product, as it should be. This is fixed in the Hartree-Fock approximation where the equation for $\varphi_i(\vec{r})$ becomes

$$H_{\mathrm{Hartree}}\varphi_i(\vec{r}) - \sum_{j \neq i} \frac{e^2}{8\pi} \int d^3 r' \frac{\varphi_j^*(\vec{r}')\varphi_i(\vec{r}')}{|\vec{r} - \vec{r}'|} \delta_{s_i s_j} \varphi_j(\vec{r}) = \varepsilon_i \varphi_i(\vec{r})$$

where s stands for spin of the electron. As to the new term on the LHS, called the exchange term, it is definitely not easy to understand intuitively. In this case, a formal derivation is perhaps more elucidatory.

A sketch of the formal derivation of the Hartree and Hartree-Fock equations looks like this: one takes the product or the anti-symmetrized product (often called the Slater determinant) in the Hartree and Hartree-Fock approximations respectively. Then one use this (anti-symmetrized product) as an ansatz for the variational method of finding (approximately) the ground state of the system, i.e. one has to minimize the matrix element $\int \psi^* H_e \psi$ with constraints $\int \varphi_i^* \varphi_i = 1$. The output of this procedure are the above equations (with ε_i entering as the Lagrange multiplicators).

The main disadvantage of the formal derivation based on the variational method is that there is no systematic way of going beyond this approximation. An alternative derivation, allowing for the clear identification of what has been omitted, is therefore most welcome. Fortunately, there is such a derivation. It is based on the Feynman diagram technique. When applied to the system of electrons with the Coulomb interaction, the diagram technique enables one to identify a subset of diagrams which, when summed together, provides precisely the Hartree-Fock approximation. This, however, is not of our concern here.

In the QM of molecules, the Hartree-Fock approximation is of the highest importance both qualitatively (it sets the language) and quantitatively (it enters detailed calculations). In the

solid state physics, the main role of the approximation is qualitative. Together with the Bloch theorem, it leads to one of the cardinal notions of this branch of physics, namely to the Fermi surface.

Bloch theorem reveals the generic form of eigenfunctions of the Hamiltonian which is symmetric under the lattice translations. The eigenfunctions are labeled by two quantum numbers n and $\vec{k}$, the former (called the zone number) being discrete and the latter (called the quasi-momentum) being continuous and discrete for infinite and finite lattices respectively (for large finite lattice, the $\vec{k}$ is so-called quasi-continuous). The theorem states that the eigenfunctions are of the form $\varphi_{n,\vec{k}}(\vec{r}) = e^{i\vec{k}.\vec{r}}u_{n,\vec{k}}(\vec{r})$, where the function $u_{n,\vec{k}}(\vec{r})$ is periodic (with the periodicity of the lattice)[2]. In the solid state physics the theorem is supplemented by the following more or less plausible assumptions:

• The theorem holds not only for the Schrödinger equation, but even for the Hartree and Hartree-Fock equations with periodic *external* potentials.

• The energy $\varepsilon_n(\vec{k})$ is for any n a (quasi-)continuous function of $\vec{k}$.

• The relation $\varepsilon_n(\vec{k}) = \text{const}$ defines a surface in the $\vec{k}$-space.

The very important picture of the zone structure of the electron states in solids results from these assumptions. So does the notion of the Fermi surface, one of the most crucial concepts of the solid state physic. In the ground state of the system of non-interacting fermions, all the one-fermion levels with energies below and above some boundary energy are occupied and empty respectively. The boundary energy is called the Fermi energy ε_F and the surface corresponding to $\varepsilon_n(\vec{k}) = \varepsilon_F$ is called the Fermi surface.

The reason for the enormous importance of the Fermi surface is that all the low excited states of the electron system lie in the vicinity of the Fermi surface. The geometry of the Fermi surface therefore determines a great deal of the physical properties of the given solid.

Calculation of the Fermi surface from the first principles is only possible with some (usually quite radical) simplifying assumptions. Surprisingly enough, yet the brutally looking simplifications use to give qualitatively and even quantitatively satisfactory description of many phenomena. However, the most reliable determinations of the Fermi surface are provided not by the theory, but rather by the experiment. There are several methods of experimental determination of the Fermi surface. The mutual consistency of the results obtained by various methods serves as a welcome cross-check of the whole scheme.

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[2]The proof is quite simple. The symmetric Hamiltonian commutes with any translation $T_{\vec{R}}$ where $\vec{R}$ is a lattice vector. Moreover, any two translations commutes with each other. Let us consider common eigenfunctions $\varphi(\vec{r})$ of the set of mutually commuting operators presented by the Hamiltonian and the lattice translations. We denote by λ_i the eigenvalues of the translations along the three primitive vectors of the lattice , i.e.$T_{\vec{a}_i}\varphi(\vec{r}) = \lambda_i\varphi(\vec{r})$. The composition rule for the translations ($T_{\vec{A}+\vec{B}} = T_{\vec{A}}T_{\vec{B}}$) implies $T_{\vec{R}}\varphi(\vec{r}) = \lambda_1^{n_1}\lambda_2^{n_2}\lambda_3^{n_3}\varphi(\vec{r})$ for $\vec{R} = \sum_{i=1}^{3} n_i\vec{a}_i$. On the other hand, $T_{\vec{R}}\varphi(\vec{r}) = \varphi(\vec{r}+\vec{R})$. This requires $|\lambda_i| = 1$, since otherwise the function $\varphi(\vec{r})$ would be unbounded or disobeying the Born-Karman periodic boundary conditions for the infinite and finite lattice respectively. Denoting $\lambda_i = e^{ik_ia_i}$ (no summation) one has $T_{\vec{R}}\varphi(\vec{r}) = e^{i\vec{k}.\vec{R}}\varphi(\vec{r})$. At this point one is practically finished, as it is now straightforward to demonstrate that $u_{\vec{k}}(\vec{r}) = e^{-i\vec{k}.\vec{r}}\varphi(\vec{r})$ is periodic. Indeed $T_{\vec{R}}e^{-i\vec{k}.\vec{r}}\varphi(\vec{r}) = e^{-i\vec{k}.(\vec{r}+\vec{R})}\varphi(\vec{r}+\vec{R}) = e^{-i\vec{k}.\vec{r}}e^{-i\vec{k}.\vec{R}}T_{\vec{R}}\varphi(\vec{r}) = e^{-i\vec{k}.\vec{r}}\varphi(\vec{r})$.

For an infinite lattice, $\vec{k}$ can assume any value ($\vec{k}$ is a continuous). For finite lattice the allowed values are restricted by the boundary conditions to discrete possible values, nevertheless for large lattices the $\vec{k}$-variable turns out to be quasi-continuous (for any allowed value, the closest allowed value is "very close"). For a given $\vec{k}$, the Hamiltonian can have several eigenfunctions and eigenvalues and so one needs another quantum number n in addition to the quantum number $\vec{k}$.

Harmonic approximation describes the system of nuclei or any other system in the vicinity of its stable equilibrium. It is one of the most fruitful approximations in the whole physics.

For the sake of notational convenience, we will consider the one-dimensional case. Moreover from now on, we will denote by R the equilibrium position of the particular nucleus, rather than the actual position, which will be denoted by $R+\rho$. The Hamiltonian of the system of nuclei $H_N = -\sum_n \frac{\Delta_n}{2M} + \mathcal{U}(R+\rho)$, where $\mathcal{U}(R+\rho) = U(R+\rho) + \bar{\varepsilon}(R+\rho)$. The potential energy $\mathcal{U}(R+\rho)$ is expanded around the equilibrium $\mathcal{U}(R+\rho) = \mathcal{U}(R) + \sum_n \mathcal{U}_n\rho_n + \sum_{m,n}\mathcal{U}_{mn}\rho_m\rho_n + \dots$ where $\mathcal{U}_n = \frac{\partial}{\partial\rho_n}\mathcal{U}(R+\rho)|_{\rho=0}$ and $\mathcal{U}_{mn} = \frac{\partial^2}{\partial\rho_m\partial\rho_n}\mathcal{U}(R+\rho)|_{\rho=0}$. The first term in the expansion is simply a constant, which can be dropped out safely. The second term vanishes because of derivatives $\mathcal{U}_i$ vanishing at the minimum of potential energy. When neglecting the terms beyond the third one, we obtain the harmonic approximation

$$H_N = -\sum_n \frac{\Delta_n}{2M} + \sum_{m,n}\mathcal{U}_{mn}\rho_m\rho_n$$

where $\Delta_n = \frac{\partial^2}{\partial\rho_n^2}$ and the matrix $\mathcal{U}_{mn}$ of the second derivatives is symmetric and positive (in the stable equilibrium the potential energy reaches its minimum).

Now comes the celebrated move: the symmetric matrix $\mathcal{U}_{mn}$ can be diagonalized, i.e. there exists an orthogonal matrix $\mathcal{D}$ such that $K \equiv \mathcal{D}\mathcal{U}\mathcal{D}^{-1}$ is diagonal ($K_{mn} = K_n\delta_{mn}$ no summation). Utilizing the linear combinations $\xi_m = \sum_n \mathcal{D}_{mn}\rho_n$, one may rewrite the Hamiltonian to the extremely convenient form

$$H_N = -\sum_n \frac{\Delta_n}{2M} + \frac{1}{2}K_n\xi_n^2$$

where now $\Delta_n = \frac{\partial^2}{\partial\xi_n^2}$ is understood[3]. This is the famous miracle of systems in the vicinity of their stable equilibriums: any such system is well approximated by the system of coupled harmonic oscillators which, in turn, is equivalent to the system of *decoupled* harmonic oscillators.

Stationary states of the system of the independent harmonic oscillators are characterized by the sequence $(n_1, n_2, \dots)$ where n_i defines the energy level of the i-th oscillator Energies of individual oscillators are $\hbar\omega_n(N_n + 1/2)$ where $\omega_n = \sqrt{K_n/M}$. Energy of the system is $\sum_i \hbar\omega_n(N_n+1/2)$. This brings us to yet another equivalent description of the considered system, namely to the (formal) ideal gas description.

Let us imagine a system of free particles, each of them having energy eigenstates labeled by n with eigenvalues $\hbar\omega_n$. If there are N_n particles in the n-th state, the energy of the system will be $\sum_n \hbar\omega_n N_n$. This is equivalent (up to a constant) to the Hamiltonian of independent oscillators. It is common habit to describe a system of independent harmonic oscillators in terms of the equivalent system of free formal particles. These formal particles are called *phonons*.

Phonons are would-be particles widely used for the formal description of real particles, namely independent harmonic oscillators (see the previous paragraph). Phonon is not a kind of particle. Strictly speaking, it is just a word.

Some people may insist on the use of the word *phonon* only for systems with some translational symmetry, rather than in connection with any system of independent harmonic oscillators. This, however, is just the matter of taste. Anyway, transitionally invariant lattices are of the

[3]This is indeed possible since $\frac{\partial}{\partial\xi_i} = \sum_j \mathcal{D}_{ij}\frac{\partial}{\partial\rho_j}$ and $\frac{\partial^2}{\partial\xi_i\partial\xi_k} = \sum_{j,l}\mathcal{D}_{ij}\frac{\partial}{\partial\rho_j}\mathcal{D}_{kl}\frac{\partial}{\partial\rho_l}$. So if one sets $k=i$ and then sums over i utilizing symmetry and orthogonality of $\mathcal{D}$ (leading to $\sum_i \mathcal{D}_{ij}\mathcal{D}_{il} = \sum_i \mathcal{D}_{ji}\mathcal{D}_{il} = \delta_{jl}$) one obtains $\sum_i \frac{\partial^2}{\partial\xi_i^2} = \sum_j \frac{\partial}{\partial\rho_j^2}$ as claimed.

utmost importance in solid state physics and, interestingly enough, in the case of a transitionally symmetric lattice (of stable equilibrium positions of individual nuclei) the diagonalizing matrix is explicitly known. Indeed, the Fourier transformation

$$\xi_m = \sum_n e^{ik_m R_n} \rho_n$$

i.e. $\mathcal{D}_{mn} = e^{ik_m R_n}$ does the job[4]. For a finite lattice with N sites and periodic boundary conditions[5], one has $k_m = \frac{2\pi}{a}\frac{m}{N}$ where a is the lattice spacing. Since there is one to one correspondence between m and k_m, one may label individual independent oscillators by k_m rather than by m. In this notation, the energies of individual oscillators become $\omega(k_m)$. For an infinite lattice, k may assume any value from the interval of the length $\frac{2\pi}{a}$, say $(-\frac{\pi}{a}, \frac{\pi}{a})$, and the corresponding energies form a function $\omega(k)$.

Three-dimensional case is treated in the same way, even if with more cumbersome notation. Phonons are characterized by the continuous and quasi-continuous $\vec{k}$ for infinite and large finite lattices respectively, and moreover by the polarization σ and by the function $\omega(\vec{k}, \sigma)$.

As for the determination of the function $\omega(\vec{k}, \sigma)$, very much the same as for the Fermi surface can be said. Namely: Calculation of $\mathcal{U}(\vec{R}+\vec{\rho})$ and consequently of $\omega(\vec{k}, \sigma)$ from the first principles is only possible with some (usually quite radical) simplifying assumptions. Surprisingly enough, yet the brutally looking simplifications use to give qualitatively and even quantitatively satisfactory description of many phenomena. However, the most reliable determinations of $\omega(\vec{k}, \sigma)$ are provided not by the theory, but rather by the experiment. There are several methods of experimental determination of $\omega(\vec{k}, \sigma)$. The mutual consistency of the results obtained by various methods serves as a welcome cross-check of the whole scheme.

[4]The proof goes very much like the one for the Bloch theorem. First, one may want to recall that the eigenvectors of the matrix $\mathcal{U}$ can be used for the straightforward construction of the diagonalizing matrix $\mathcal{D}$ (the matrix with columns being the eigenvectors of the matrix $\mathcal{U}$ is the diagonalizing one, as one can verify easily). Then one utilizes the invariance of $\mathcal{U}$ with respect to the lattice translations T_{na} ($T_{na}\rho_j = \rho_{j+n}$) which implies $[\mathcal{U}, T_{na}] = 0$. As to the common eigenvectors (of $\mathcal{U}$ and all T_{na}), if one has $\forall j\ T_a\rho_j = \lambda\rho_j$ then $\forall j\ T_{na}\rho_j = \lambda^n\rho_j$ (since $T_{na} = (T_a)^n$) and this implies $\rho_{j+n} = \lambda^n\rho_j$. Denoting the m-th eigenvalue $\lambda_m = e^{ik_m a}$ one obtains $\rho_n = c.e^{ik_m na} = c.e^{ik_m R_n}$.

It is perhaps worth mentioning that the above explanation of the origin of the Fourier transformation in these circumstances seems to be somehow unpopular. Most authors prefer to introduce it other way round. They discuss the system of *classical* rather than quantum harmonic oscillators, in which case the Fourier transformation comes as a natural tool of solving the system of coupled ordinary differential equations. The resulting system of independent classical oscillators is canonically quantized afterwards. Nevertheless, there is no need for this side-step to the classical world. The problem is formulated at the quantum level from the very beginning and it can be solved naturally entirely within the quantum framework, as we have demonstrated.

[5]The postulate of the periodic boundary conditions is just a convenient technical device rather than a physically sound requirement.

The basic approximations and the a^+, a operators

Both the Hartree-Fock and the harmonic approximations lead to the systems of ideal gases. Their Hamiltonians can be therefore rewritten in terms of the creation and annihilation operators easily. For the electron system one has

$$H_e = \sum_{n,\sigma} \int d^3k \; \varepsilon_n(\vec{k}) \; a^+_{n,\vec{k},\sigma} a_{n,\vec{k},\sigma}$$

Using the electron-hole formalism (see p. 26) this is commonly rewritten as

$$H_{e,h} = -\sum_{n,\sigma} \int d^3k \; \varepsilon^h_n(\vec{k}) \; b^+_{n,\vec{k},\sigma} b_{n,\vec{k},\sigma} + \sum_{n,\sigma} \int d^3k \; \varepsilon^e_n(\vec{k}) \; a^+_{n,\vec{k},\sigma} a_{n,\vec{k},\sigma} + \text{const}$$

where $\varepsilon^e_n(\vec{k})$ and $\varepsilon^h_n(\vec{k})$ vanish beneath and above the Fermi surface respectively (they equal to $\varepsilon_n(\vec{k})$ otherwise). For the nucleus system one has (cf. p. 23)

$$H_N = \sum_{\sigma} \int d^3k \; \hbar\omega(\vec{k},\sigma) \; A^+_{\vec{k},\sigma} A_{\vec{k},\sigma}$$

Note that one can write the Hamiltonian of the system of electrons and nuclei with Coulomb interactions in terms of the creation and annihilation operators of electrons and nuclei (we have learned how to do this in previous sections). The outcome of the basic approximation differs from this straightforward use of the creation and annihilation operators in many respects.

- Both the electron-hole and the phonon systems do not have a fixed number of particles. They contain, e.g., no particles in the ground states of the electron-hole and the harmonic approximations respectively. The low excited states are characterized by the presence of some "above Fermi" electrons and holes and/or few phonons. The number of the genuine electrons and nuclei, on the other hand, is fixed.

- There are no interactions between the Hartree-Fock electrons and holes and the phonons. The interactions between the genuine electrons and nuclei, on the other hand, are absolutely crucial.

- There are no unknown parameters or functions in the Hamiltonian in terms of creation and annihilation operators of the genuine electrons and nuclei. The Hamiltonian of both ideal gases of the Hartree-Fock electrons and holes and of the phonons do contain such unknown functions, namely the energies of individual one-particle states. These usually need to be determined phenomenologically.

Beyond the basic approximations

The formalism of creation and annihilation operators is useful even beyond the basic approximations. Let us briefly mention some examples.

Beyond the Hartree-Fock approximation there is always some (educated-guessed, model-dependent) potential energy between Hartree-Fock electrons and holes. It is straightforward to write it in terms of the electron-hole creation and annihilation operators. Application: excitons (a bound states of an electron and a hole).

Beyond the harmonic approximation there are higher (than quadratic) terms in the Taylor expansion of the potential (they were neglected in the harmonic approximation). To write them in terms of the phonon creation and annihilation operators, one has to write the position operator, i.e. the operator of the deviation $\vec{\rho}$ of nuclei from the equilibrium position in terms of the phonon variable $\vec{\xi}$ and then to write the $\vec{\xi}$-operator in terms of $A^+_{\vec{k},\sigma} A_{\vec{k},\sigma}$ (which is just a notorious LHO relation). The result is the phonon-phonon interaction. Application: thermal expansion and thermal conductivity.

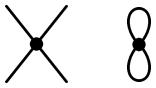
Beyond the adiabatic approximation there is some (educated-guessed, model-dependent) potential energy between Hartree-Fock electrons and nuclei. This is written in terms of electron and nucleus creation and annihilation operators. The potential energy is Taylor expanded in nuclei position deviations $\vec{\rho}$ and the first term (others are neglected) is re-expressed in terms of phonon operators. The resulting intearction is the electron-phonon interaction. Application: Cooper pairs in superconductors (a bound states of two electrons).

pred cross-sections and decay rates

connected diagrams

IIThe M_{fi}, or rather m_{fi} within our naive approach, is more appropriate for the discussion of the cross-sections and decay rates, which is our next task.At this point, there are only three differences left:

- presence of $d_F(p)$ instead of the genuine Feynman propagator $D_F(p)$
- no $\sqrt{Z}$ factors corresponding to external legs
- presence of disconnected diagrams like



in the perturbative expansions of the green function and the s-matrix[6], while in the Introductions/Conclusions only connected Feynman diagrams were accounted for.

The differences are due to the fact that we are dealing with the s-matrix rather than the S-matrix. In the next section we will learn how the so-far missed ingredients (replacement of d_F by D_F, appearance of $\sqrt{Z}$ and fadeaway of disconnected diagrams) will enter the game in the standard approach.

As to the comparison of the rules presented in the Introductions/Conclusions to the ones derived here, let us remark that in the Introductions/Conclusions we did not introduce the notion of the S-matrix explicitly. Neverthweless, it was present implicitly via the quantity M_{fi}, since S and M are very closely related

$$S_{fi} = \mathbf{1} + iM_{fi}(2\pi)^4 \delta^{(4)}(P_f - P_i)$$

The M_{fi}, or rather m_{fi} within our naive approach, is more appropriate for the discussion of the cross-sections and decay rates, which is our next task.

Remark: *As we have seen, the green functions g and the s-matrix elements s_{fi} are very closely related. The only differences are the external legs factors: $d_F(p)$ for the green function g and simply 1 (or something slightly more complicated in case of higher spins) for the s-matrix elements. This may be formulated in the following way: s_{fi} is obtained from the corresponding green function g by multiplication of each external leg by the inverse propagator. Another, even more popular, formulation: s_{fi} is obtained from the corresponding g by amputation of the external legs. Actually, the relation between s and g is of virtually no interest whatsoever. We are, after all, interested only in the s-matrix, so there is no reason to bother about the green functions. Indeed, we could ignore the whole notion of the green function and derive the rules directly for the s-matrix. Doing so, however, we would miss the nice analogy between the naive and the standard approaches.*
The point is that similar relation holds also for the genuine Green functions G and the S-matrix elements. Indeed, as we will see, S_{fi} is obtained from the corresponding Green function G by amputation of the external leg and multiplication by $\sqrt{Z}$. And in the standard approach, unlike in the naive one, one cannot easily avoid the Green functions when aiming at the S-matrix.

[6]Which term in the perturbative expansion of $U(T,-T)$ corresponds this diagram to?

remark on complex fields and arrows

The expansion of the complex scalar field in creation and annihilation operators reads

$$\varphi_I(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}(t)\,e^{i\vec{p}.\vec{x}} + b^+_{\vec{p}}(t)\,e^{-i\vec{p}.\vec{x}}\right)$$
$$\varphi^*_I(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a^+_{\vec{p}}(t)\,e^{-i\vec{p}.\vec{x}} + b_{\vec{p}}(t)\,e^{i\vec{p}.\vec{x}}\right)$$

with

$$\left[a_{\vec{p}}(t), a^+_{\vec{p}'}(t)\right] = (2\pi)^3\,\delta(\vec{p}-\vec{p}')$$
$$\left[b_{\vec{p}}(t), b^+_{\vec{p}'}(t)\right] = (2\pi)^3\,\delta(\vec{p}-\vec{p}')$$

and all other commutators of creation and annihilation operators equal to zero. It is now straightforward to show that

$$T\{\varphi_I(x)\,\varphi_I(y)\} = N\{\varphi_I(x)\,\varphi_I(y)\}$$
$$T\{\varphi^*_I(x)\,\varphi^*_I(y)\} = N\{\varphi^*_I(x)\,\varphi^*_I(y)\}$$
$$T\{\varphi_I(x)\,\varphi^*_I(y)\} = N\{\varphi_I(x)\,\varphi^*_I(y)\} + d_F(x-y)$$
$$T\{\varphi^*_I(x)\,\varphi_I(y)\} = N\{\varphi^*_I(x)\,\varphi_I(y)\} + d_F(x-y)$$

This means that the time ordered product of two φ_I-fields as well as of the two φ^*_I-fields is already in the normal form, i.e. the only contributions to the Feynman diagrams come from $T\{\varphi_I\varphi^*_I\}$ and $T\{\varphi^*_I\varphi_I\}$.

This result is typical for complex fields: they provide two types of propagators, corresponding to products of the field and the conjugate field in two possible orderings. In case of the complex scalar field the factors corresponding to the two different orderings are equal to each other, so there is no reason to use two different graphical representations. This, however, is not a general feature. In other cases (e.g. in case of the electron-positron field) the different orderings lead to different factors. It is therefore necessary to distinguish these two possibilities also in their graphical representation, and this is usually done by means of an arrow.