Double Pendulum

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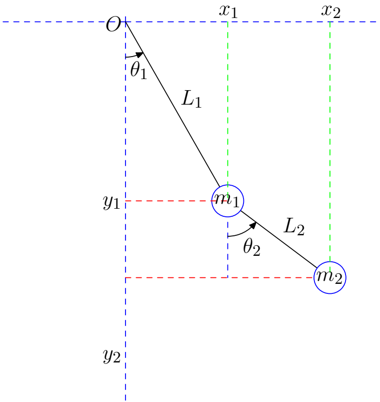
Position

$$x_1 = L_1 \sin(\theta_1)$$
$$x_2 = L_1 \sin(\theta_1) + L_2 \sin(\theta_2)$$
$$y_1 = -L_1 \cos(\theta_1)$$
$$y_2 = -L_1 \cos(\theta_1) - L_2 \cos(\theta_2)$$



Potential Energy: the sum of the potential energy of each mass

$$P = m_1 g y_1 + m_2 g y_2$$

$$P = -m_1 g L_1 \cos(\theta_1) - m_2 g \left(L_1 \cos(\theta_1) + L_2 \cos(\theta_2) \right)$$

Kinetic Energy in General

We know that

$$K = 1/2mv^2.$$

Which brings us to

$$K = 1/2m(\dot{x}^2 + \dot{y}^2).$$

Kinetic Energy in the double pendulum system

$$K = 1/2 m_1(\dot{x}_1^2 + \dot{y}_1^2) + 1/2 m_2(\dot{x}_2^2 + \dot{y}_2^2).$$

position:

$$x_1 = L_1 \sin(\theta_1)$$
$$x_2 = L_1 \sin(\theta_1) + L_2 \sin(\theta_2)$$
$$y_1 = -L_1 \cos(\theta_1)$$
$$y_2 = -L_1 \cos(\theta_1) - L_2 \cos(\theta_2)$$

differentiating:

$$\dot{x}_1 = L_1 \cos(\theta_1)\dot{\theta}_1$$
$$\dot{x}_2 = L_1 \cos(\theta_1)\dot{\theta}_1 + L_2 \cos(\theta_2)\dot{\theta}_2$$
$$\dot{y}_1 = L_1 \sin(\theta_1)\dot{\theta}_1$$
$$\dot{y}_2 = L_1 \sin(\theta_1)\dot{\theta}_1 + L_2 \sin(\theta_2)\dot{\theta}_2$$

$$K = 1/2 m_1 \dot{\theta}_1^2 L_1^2 + 1/2 m_2[\dot{\theta}_1^2 L_1^2 + \dot{\theta}_2^2 L_2^2 + 2\dot{\theta}_1 L_1 \dot{\theta}_1 L_2 \cos(\theta_1 - \theta_2)].$$

Lagrangian in General

The Lagrangian(L) of a system is defined to be the difference of the kinetic energy and the potential energy.

$$L = K - P.$$

For the Lagrangian of a system this Euler-Lagrange differential equation must be true:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

the Lagrangian of our double pendulum system

$$K = 1/2 m_1 \dot{\theta}_1^2 L_1^2 + 1/2 m_2 [\dot{\theta}_1^2 L_1^2 + \dot{\theta}_2^2 L_2^2 + 2\dot{\theta}_1 L_1 \dot{\theta}_2 L_2 \cos(\theta_1 - \theta_2)].$$

$$P = -(m_1 + m_2) g L_1 \cos(\theta_1) - m_2 L_2 g \cos(\theta_2)$$

In our case the Lagrangian is

$$L = 1/2 (m_1 + m_2) L_1^2 \dot{\theta}_1^2 + 1/2 m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 + \theta_2) + (m_1 + m_2) g L_1 \cos(\theta_1) + m_2 L_2 g \cos(\theta_2).$$

Partials of the Lagrangian for θ_1

$$L = 1/2(m_1 + m_2)L_1^2\dot{\theta}_1^2 + 1/2 m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g L_1 \cos(\theta_1) + m_2 L_2 g \cos(\theta_2)$$

Thus:

$$\frac{\partial L}{\partial \theta_1} = -L_1 g (m_1 + m_2) \sin(\theta_1) - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) L_1^2 \dot{\theta}_1 + m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 L_1 L_2 \dot{\theta}_2 sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

Substituting into the Euler-Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$(m_1 + m_2)L_1^2\ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 L_1 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + gL_1(m_1 + m_2)\sin(\theta_1) = 0$$

Simplifying and Solving for $\ddot{\theta}_1$:

$$\ddot{\theta}_1 = \frac{-m_2 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g \sin(\theta_1)}{(m_1 + m_2) L_1}$$

Partials for θ_2

Once again the Lagrangian of the system is

$$L = 1/2(m_1 + m_2)L_1^2\dot{\theta}_1^2 + 1/2m_2L_2^2\dot{\theta}_2^2 + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + (m_1 + m_2)gL_1\cos(\theta_1) + m_2L_2g\cos(\theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - L_2m_2g\sin(\theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2L_2^2\dot{\theta}_2 + m_2L_1L_2\dot{\theta}_1\cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2L_2^2\ddot{\theta}_2 + m_2L_1L_2\ddot{\theta}_1\cos(\theta_1 - \theta_2) - m_2L_1L_2\dot{\theta}_1\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

Substituting into the Euler-Lagrange equation for θ_2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$L_2\ddot{\theta}_2 + L_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) - L_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + g\sin(\theta_2) = 0.$$

$$\ddot{\theta}_2 = \frac{-L_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) + L_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - g\sin(\theta_2)}{L_2}.$$

two dependent differential equations

We now have two equations that both have $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in them.

$$\ddot{\theta}_1 = \frac{-m_2 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g \sin(\theta_1)}{(m_1 + m_2) L_1}$$

$$\ddot{\theta}_2 = \frac{-L_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + L_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - g \sin(\theta_2)}{L_2}.$$

creating two second order differential equations

$$\ddot{\theta}_1 = \frac{\begin{array}{r}-m_2 L_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) + g m_2 \sin(\theta_2)\cos(\theta_1 - \theta_2) \\ -m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g \sin(\theta_1)\end{array}}{L_1(m_1 + m_2) - m_2 L_1 \cos^2(\theta_1 - \theta_2)}$$

$$\ddot{\theta}_2 = \frac{\begin{array}{c}m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) + g \sin(\theta_1)\cos(\theta_1 - \theta_2)(m_1 + m_2) \\ + L_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)(m_1 + m_2) - g \sin(\theta_2)(m_1 + m_2)\end{array}}{L_2(m_1 + m_2) - m_2 L_2 \cos^2(\theta_1 - \theta_2)}$$

converting to a system of first order differential equations

If I define new variables for θ_1,$\dot{\theta}_1$,θ_2 and $\dot{\theta}_2$ I can construct a system of four first order differential equations that I can then solve numerically.

This gives me

$$
\begin{aligned}
z_1 &= \theta_1 \\
z_2 &= \theta_2 \\
z_3 &= \dot{\theta}_1 \\
z_4 &= \dot{\theta}_2.
\end{aligned}
$$

differentiating I get

$$
\begin{aligned}
\dot{z}_1 &= \dot{\theta}_1 \\
\dot{z}_2 &= \dot{\theta}_2 \\
\dot{z}_3 &= \ddot{\theta}_1 \\
\dot{z}_4 &= \ddot{\theta}_2.
\end{aligned}
$$

A system of four first order differential equations

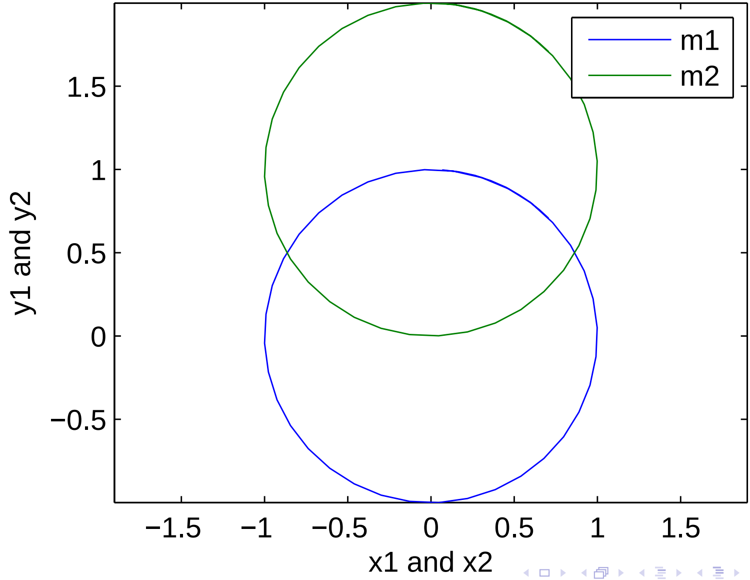
$$\dot{z}_1 = \dot{\theta}_1$$

$$\dot{z}_2 = \dot{\theta}_2$$

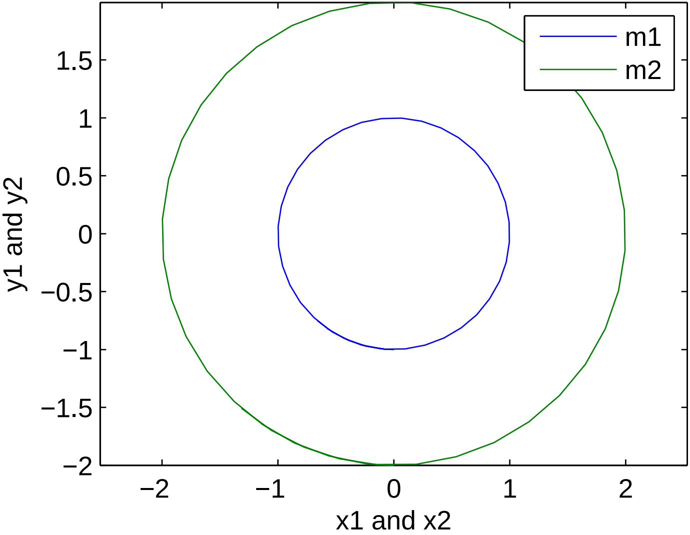
$$\dot{z}_3 = \frac{\begin{array}{r} -m_2 L_1 z_4^2 \sin(z_1 - z_2)\cos(z_1 - z_2) + g m_2 sin(z_2)\cos(z_1 - z_2) \\ -m_2 L_2 z_4^2 \sin(z_1 - z_2) - (m_1 + m_2) g \sin(z_1) \end{array}}{L_1(m_1 + m_2) - m_2 L_1 \cos^2(z_1 - z_2)}.$$

$$\dot{z}_4 = \frac{\begin{array}{r} m_2 L_2 z_4^2 \sin(z_1 - z_2)\cos(z_1 - z_2) + g\sin(z_1)\cos(z_1 - z_2)(m_1 + m_2) \\ +L_1 z_4^2 \sin(z_1 - z_2)(m_1 + m_2) - g\sin(z_2)(m_1 + m_2) \end{array}}{L_2(m_1 + m_2) - m_2 L_2 \cos^2(z_1 - z_2)}.$$

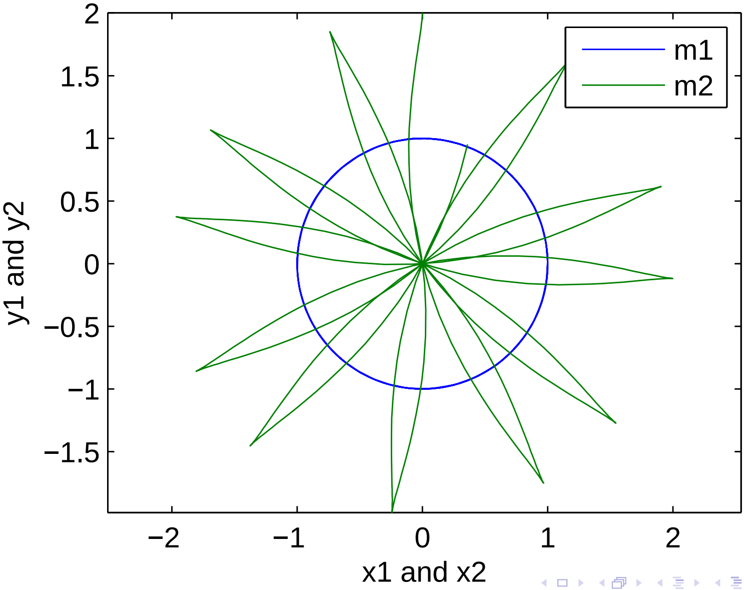
example of cyclical behavior of the system



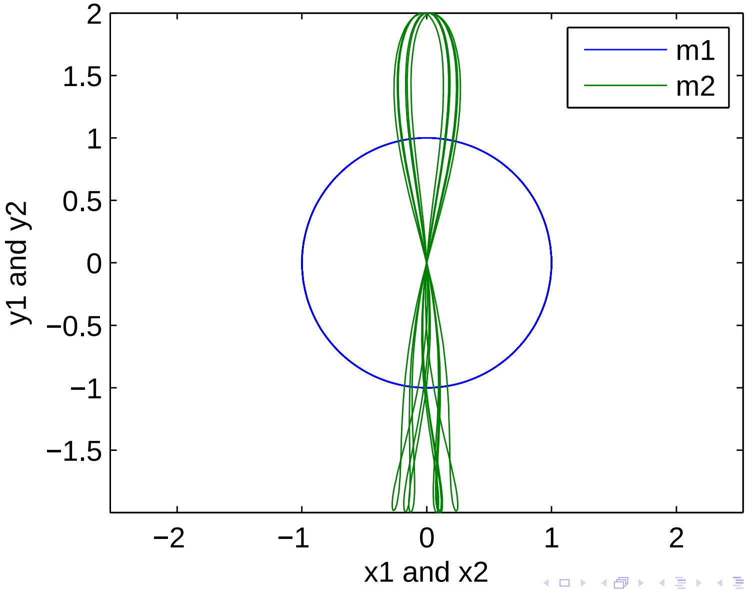
example of cyclical behavior of the system



example of nearly cyclical behavior of the system



example of nearly cyclical behavior of the system



Example of Chaotic behavior of the system