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On a geometrical formulation of the Nambu dynamics

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A geometrical formulation of the N-triplet Nambu dynamics is given. It enables one to treat some general issues of the latter, including integral invariants and canonical transformations in a simple way. A possible underlying geometrical structure is discussed.

I. INTRODUCTION

In Ref. 1 Nambu presented an interesting modification of the classical Hamiltonian dynamics. In his approach a three-dimensional phase space with the coordinates \( r = (x, y, z) \) is introduced ("Nambu triplet"), with the dynamics given by the system of first-order differential equations

\[
\dot{r} = \nabla H \times \nabla G, \quad (1)
\]

\( H(r), G(r) \) being arbitrary functions on phase space. Equivalently, one can write

\[
f = \frac{\partial (f, H, G)}{\partial (x, y, z)} + \frac{\partial f}{\partial t}, \quad (2)
\]

for any function on the "extended" phase space \( f(r,t) \). Thus the time evolution is governed by two "Hamiltonians," \( H, G \), as opposed to a single one, \( H \), in the Hamiltonian case. He showed that the Euler equations for a free rigid rotator take just this form if one identifies \((x, y, z)\) with the body-system projections of the angular momentum and \( H \) and \( G \) with the two conserved quantities available in this case, \( v_i \), the energy and the square of the angular momentum. He demonstrated that the dynamical systems (see Ref. 2 for some other examples) given by (1) share some important properties with the Hamiltonian ones, in spite of being obviously non-Hamiltonian (the dimension of the phase space is odd). In particular, the Liouville theorem holds (such "Liouville" systems were studied later in Ref. 3). He also proposed the natural generalization to more "triplets":

\[
\dot{r}_a = \nabla H \times \nabla e_a G, \quad a = 1, \ldots, N, \quad (3)
\]

or, equivalently,

\[
f = \sum_{a=1}^{N} \frac{\partial (f, H, G)}{\partial (x_ay_aw_a)} + \frac{\partial f}{\partial t}, \quad f = f(x_1, \ldots, x_N, t). \quad (4)
\]

Later on, several different aspects of the dynamics were studied by many authors. \(^4\)

In this paper, we focus our attention on a geometrical formulation of Eqs. (3), since it enables one to study the more advanced issues of Nambu dynamics in a simple way.

In Ref. 5 (published almost immediately after Ref. 1), the first attempt in this direction was made. As is well known, \(^6\) the geometrical version of the Hamiltonian equations reads (for the autonomous system)

\[
\Gamma \cdot \omega = -dH, \quad (5)
\]

where \( \omega \) is symplectic form and \( \Gamma \) is the dynamical vector field. The standard form

\[
\dot{q}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q^a}, \quad (6)
\]

is obtained using the canonical coordinates \( q^a, p_a \) (guaranteed via the Darboux theorem) in which

\[
\omega = dp_1 \land dq_1 + \cdots + dp_N \land dq_N. \quad (7)
\]

In full analogy, the three-form

\[
\omega = dx_1^1 \land dx_1^2 \land dx_1^3 + \cdots + dx_N^1 \land dx_N^2 \land dx_N^3 \quad (8)
\]

is introduced [it is closed and has maximum rank just as does (7)] in Ref. 5 and the equation

\[
\Gamma \cdot \omega = dH \land dG \quad (9)
\]

is claimed to be equivalent to (3) and thus to represent its geometrical version.

The first hint that it cannot be the case comes from the observation (made in Ref. 5) that the Liouville theorem does not hold for \( N > 1 \) for (9). Indeed, although

\[
\epsilon_1 \omega = 0 \quad (10)
\]

is a simple consequence of (9), the \( 3N \)-form \( \omega \land \cdots \land \omega \) [the only candidate for a volume form constructed from the elements occurring in (9)] vanishes identically (because of the properties of \( \land \)). This means that all but the lowest "Poincaré-Cartan" integral invariants vanish identically. In particular, there is no \( \Gamma \)-invariant volume
form for (9); i.e., the Liouville theorem does not hold (for $N > 1$) for (9).

This is, however, in contradiction to the opposite assertion [concerning, however, (3)] in Ref. 1, and one can simply verify by "traditional" means that the dynamical field in (3) actually is divergenceless with respect to the natural volume form [cf. (27)]

$$\Omega = dx^1 \wedge \cdots \wedge dx^N$$  \hspace{1cm} (11)

Indeed, if $\Gamma \equiv \Gamma_1 + \cdots + \Gamma_N \equiv \nabla_a H \times \nabla_a G$, then

$$\text{div} \Omega = \nabla_1 \Gamma_1 + \cdots + \nabla_N \Gamma_N = 0,$$  \hspace{1cm} (12)

and the Liouville theorem does hold [cf. (29)] in agreement with Ref. 1. This indicates that (9) cannot represent the geometrical version of (3) for $N > 1$ and it has to be changed (in the next section, we show that it is even inconsistent).

II. A GEOMETRICAL FORMULATION OF EQ. (3)

The three-form $\omega$ at a fixed point $P$ can be interpreted as a linear map:

$$\omega : T_P \rightarrow \Lambda^2 T^*_P,$$  \hspace{1cm} (13)

($T_P$ is the tangent space in $P$, $\Lambda^2 T^*_P$ is the space of two-forms in $P$). Since $\dim T_P < \dim \Lambda^2 T^*_P$ (for $N > 1$), the two-field form $\Gamma \omega$ in (9) is not "general" and cannot be therefore equated to $dH \wedge dG$ without some additional restrictions on $H$ and $G$. There are, however, no restrictions on $H, G$ in (3), or, in the coordinates used in (8), $\Gamma \omega$ does not contain products $dx_a \wedge dx_b$, while $dH \wedge dG$, in general, does. This shows that (9) is not only inequivalent to (3) for more than one triplet but is then even inconsistent for general $H, G$.

Let us define "partial" ($a$th triplet) exterior derivative operators $d_a$, $a = 1, \ldots, N$, operating in the following way: If

$$\sigma = \sigma_{j_1 \cdots j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p} \equiv \sigma_{j} dx^j$$

is any $p$-form, then let (the summation convention with respect to $i$ from 1–3 is understood but is in abeyance with respect to the index "$a" in what follows)

$$d_a \sigma := \frac{\partial \sigma_j}{\partial x^a} dx^j \wedge dx^a.$$  \hspace{1cm} (14)

Then,

$$d = d_1 + \cdots + d_N$$  \hspace{1cm} (15)

and

$$d_a d_b = d_a d_b;$$  \hspace{1cm} (16)

in particular,

$$d_a^2 = 0.$$

In addition, let us define $N$ three-forms as

$$\omega_a := dx_a \wedge dx_b \wedge dx_c.$$  \hspace{1cm} (18)

Then, a simple calculation shows that

$$\Gamma \wedge \omega_a = d_a H \wedge d_a G, \quad a = 1, \ldots, N$$  \hspace{1cm} (19)

is equivalent to (3) and therefore represents a geometrical formulation of the $N$-triplet Nambu equations.

III. SOME PROPERTIES OF THE NAMBU DYNAMICS

In this section we use the formulation (19) of the Nambu dynamics to analyze some of its more advanced issues.

A. Integral invariants

From the Hamiltonian equations, it follows immediately that

$$E_\Gamma \omega_a = 0$$  \hspace{1cm} (20)

$$\Rightarrow E_\Gamma (\omega_1 \wedge \cdots \wedge \omega_N) \equiv \prod_{a=1}^{N}\omega_{a} = 0,$$

which results in the series of the Poincaré-Cartan (absolute) integral invariants:

$$I_{2k} = \int_{M^{2k}} \omega^k, \quad k = 1, \ldots, n,$$  \hspace{1cm} (21)

$M^{2k}$ being any $2k$-dimensional domain in the phase space. Let us investigate whether similar integral invariants occur in Nambu dynamics also. We have

$$E_\Gamma \omega_a = \Gamma \wedge (d_a \omega_a) + d(\Gamma \wedge \omega_a) = 0,$$  \hspace{1cm} (22)

in general, (since $dd_a = 0$) and similarly

$$E_\Gamma \omega = \sum_{a=1}^{N} d(d_a H \wedge d_a G) 
eq 0,$$  \hspace{1cm} (23)

[\omega coincides with the three-form introduced in (8) and (9)]. This means, however, that neither

$$\int_{M} \omega_a$$  \hspace{1cm} (24)

nor

$$d_a^2 = 0.$$  \hspace{1cm} (17)
is invariant with respect to the time evolution in phase space. In the same manner, one verifies that none of the products \( \omega_a \wedge \omega_b \wedge \omega_c \wedge \cdots \), is \( \Gamma \) invariant except for the last one:

\[
\mathcal{L}_\Gamma (\omega_1 \wedge \cdots \wedge \omega_N) = 0. \tag{26}
\]

If we therefore define the volume 3N-form as

\[
\Omega := \omega_1 \wedge \cdots \wedge \omega_N = dx_1^1 \wedge \cdots \wedge dx_N^3. \tag{27}
\]

[it is up to a constant multiple the only candidate which can be constructed from \( \omega_a \) and happens to coincide with \( \omega \)], we have

\[
\mathcal{L}_\Gamma \Omega = 0, \tag{28}
\]

equivalent to the Liouville theorem

\[
I_{1N} := \int M^{3N} \Omega = \text{invariant}. \tag{29}
\]

The conclusion is then that no integral invariant exists but the highest one—the result just opposite to that given in Ref. 5 (and also different from Ref. 7, according to which even the complete series of the integral invariants \( I_3, \ldots, I_{3N} \) exists).

**B. Canonical transformations**

Let \( \Psi_\lambda \) be the one-parameter group of coordinate transformations generated by the vector field \( U \). Then (19) shows that they are canonical [preserving the structure of Eqs. (3)] iff they preserve the structure (18) of each \( \omega_a \) separately; i.e., iff

\[
\mathcal{L}_U \omega_a = 0, \quad a = 1, \ldots, N
\]

holds. Equation (30) leads to

\[
U = \sum_{a=1}^{N} U_a (x^1_a, x^2_a, x^3_a) \frac{\partial}{\partial x^a} \Rightarrow \sum_{a=1}^{N} \frac{\partial U_a}{\partial x^a} = 0 \tag{31}
\]

\((U_a \) does not depend on \( x^b_i \) for \( b \neq a \)).

This means that a general canonical transformation (homotopic to the identity) preserves the decomposition into the triplets (only the variables of the \( a \)th triplet enter the “new” \( a \)th triplet) and, in addition, it is volume preserving within each triplet:

\[
x'^a_i = f^a_i(x^1_a, x^2_a, x^3_a), \quad J(x^a_i x'^a_i) = 1, \tag{32}
\]

where \( f^a_i \) are arbitrary functions and \( J \) is the Jacobian of the transformation \( x^a_i \rightarrow x'^a_i \). Since any permutation of complete triplets

\[x^j_i \rightarrow x^j_{\sigma(i)} \]
tungished coordinates. This remains, in fact, an open problem, which can be formulated as follows: Find a tensor field such that it can live only on a 3N-dimensional manifold and it singles out naturally "canonical" coordinates occurring in (15)–(19). A manifold \( M \) endowed with a tensor field of this type should be then the natural living space for \( N \)-triplet Nambu dynamics.

As an example, the symplectic form, as is well known, can live only on an even dimensional manifold and singles out the class of canonical coordinates (via Darboux theorem).

Another useful example is when the almost complex structure on a manifold [the \((1, 1)\) type tensor field \( C \) with the property \( C^2 = -1 \)], which can also exist only on an even-dimensional manifold, induces the coordinates \( \bar{z}_1, \bar{z}_2 \) as well as the decomposition (compare with (15)–(18)) of the exterior derivative operator:

\[
d = \partial + \bar{\partial},
\]

\[
\partial(\alpha_j dx^j) := \frac{\partial \alpha_j}{\partial x^i} dx^i \wedge dx^j,
\]

\[
\bar{\partial}(\alpha_j dx^j) := \frac{\partial \alpha_j}{\partial \bar{x}^i} d\bar{x}^i \wedge dx^j,
\]

\[
d^2 = \partial^2 = \bar{\partial}^2 = 0, \quad \partial \bar{\partial} = - \bar{\partial} \partial
\]

[here, \( x \equiv (x_1, \ldots, x_n, \bar{z}_1, \ldots, \bar{z}_p) \); note that, by introducing the notations \( x_1^j \equiv z_j, x_j^i \equiv \bar{z}_i \), the formal coincidence of the corresponding formulas is obtained]. Similarly, the well-known decomposition of a \( p \)-form as the sum of \((k, l)\)-type \((k + l = p)\) forms invariant with respect to complex-analytical coordinate transformations (playing the role of the canonical transformations for \( C \)) has its evident counterpart in Nambu dynamics in the unique decomposition of a \( p \)-form into the sum of \((k_1, \ldots, k_N)\)-type forms, \( k_1 + \cdots + k_N = p \), invariant with respect to the coordinate transformations preserving triplets.

One might guess that the role of the tensor field mentioned above could be played by the maximum-rank closed three-form as opposed to the maximum-rank closed two-form (\( \equiv \) symplectic form), which serves this purpose in the Hamiltonian dynamics. If a theorem (analogous to the Darboux theorem) leading to the canonical expression (8) (and restricting thus automatically the dimension of the manifold to \( 3N \)) existed, then it would be possible to define the triplet-decomposition via (19), which was replaced by (19). Two issues were studied then. First, the question of the integral invariants was analyzed. It was shown that only the highest one, \( I_{3N} \), exists (with respect to the finite development; see remark at the end of Sec. III B). This means, in particular, that the Liouville theorem does hold for any \( N \).

Second, the canonical transformations were studied, and the analysis confirmed the results of Ref. 1.

A formulation (19) singles out a class of distinguished coordinates—a decomposition into triplets is needed. The origin of this decomposition is clear for some known concrete systems (e.g., the coupled spin system). Some speculations were made, however, concerning a possible geometrical structure leading to (19) naturally. It was shown that a maximum-rank closed three-form is not enough to serve this purpose since a straightforward generalization of the Darboux theorem to such three-forms does not hold.

\[ 8 \] M. Fecko, submitted for publication.