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A QUANTUM SPIN MODEL WITH UNSTABLE STATIONARY STATES

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An exactly solvable model of a quantum lattice system with infinite many degrees of freedom and with a finite range interaction is constructed. There are stationary states of the system without being convex combinations of KMS and ground states. It is shown that all local perturbations of such a state lead for large times $t=\infty$ to states macroscopically different from the initial ones.

СПИНОВАЯ КВАНТОВАЯ МОДЕЛЬ С НЕСТАБИЛЬНЫМИ СТАЦИОНАРНЫМИ СОСТОЯНИЯМИ

Построена точно решаемая модель квантовой решётчатой системы с бесконечным числом степеней свободы и с конечным радиусом взаимодействия. Существуют стационарные состояния системы без того, чтобы они являлись выпуклыми комбинациями КМС и основных состояний. Показано, что для больших времён ($t \to \infty$) все локальные возмущения такого состояния приводят к состояниям, макроскопически отличающимся от первоначальных.

I. INTRODUCTION

Metastable phenomena occurring in macroscopic system (e.g. overheated liquid in a bubble chamber) are not well understood from the point of view of dynamical laws of quantum mechanics. The number of microscopic degrees of freedom of systems exhibiting some kind of metastability (or irreversibility) is very large. Rigorous mathematical analysis of the behaviour of "realistic" systems consisting of many microscopic constituents is a difficult problem. A simple solvable model is constructed in this paper to demonstrate a "theoretical possibility" of dynamical description of systems with metastable states in the framework of quantum mechanics of big systems. It is a model of an infinite chain of 1/2-spins with a finite range translationally invariant interactions.

The language of a quasilocal C^*-algebra $\mathfrak{A}$ describing the system as the algebra of all its observables is used. $\mathfrak{A} \equiv \bar{\mathfrak{A}}_L$ is the norm closure of an algebra $\mathfrak{A}_L$ of local

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quantities: $\mathfrak{A}_L \equiv \bigcup_X \mathfrak{A}_X$, where $\mathfrak{A}_X$ are algebras describing finite subsystems of the system. The algebras $\mathfrak{A}_X$ are, in our case, algebras of all linear operators in finite dimensional Hilbert space. Hence, the C^*-inductive limit $\mathfrak{A}$ of the net of algebras $\mathfrak{A}_X$ is simple. The convex set of states $\mathcal{S}(\mathfrak{A}) \subset \mathfrak{A}^*$ is the set of all positive normalized linear functionals on $\mathfrak{A}$ and $\mathfrak{A}^*$ is the dual of the Banach space $\mathfrak{A}$. For $\omega \in \mathcal{S}(\mathfrak{A})$, $x \in \mathfrak{A}_L$ and $\omega(x^*x) = 1$, the state ω_x, $\omega_x(y) \equiv \omega(x^*yx)$ for $y \in \mathfrak{A}$, is called a *local perturbation* of the state ω. The time evolution of the system is described by a strongly continuous one-parameter group of *-automorphisms $\tau_t \in \operatorname{aut} \mathfrak{A}$ and the time evolution of the initial state ω is given by

$$\tau_t^* \omega(x) \equiv \omega^t(x) \equiv \omega(\tau_t x), \qquad x \in \mathrm{A},\ \omega \in \mathcal{S}(\mathfrak{A}).$$

A *stationary state* ω fulfils $\tau_t^* \omega = \omega$ for all $t \in \mathbf{R}$.

We are interested in the behaviour of states $\tau_t^* \omega_x^0 \equiv \omega_x^t$ for $t \to \infty$, where ω^0 is a certain stationary state (called here the vacuum state) and ω_x^0 $(x \in \mathfrak{A}_L)$ are its local perturbations. The state ω^0 is a stationary state, which is neither a ground state nor a KMS-state and is extremal (pure) in $\mathcal{S}(\mathfrak{A})$ (compare [3], where such states were not known). The limits w* — $\lim_{t \to \infty} \omega_x^t \equiv \bar{\omega}_x \in \mathcal{S}(\boldsymbol{A})$ for some $x \in \mathfrak{A}_L$ exist and are disjoint from ω_x^0. (All the states ω^0, ω_x^0 and ω_x^t for all $t \in \boldsymbol{R}$ are pure and mutually equivalent)[1]. The limits $\bar{\omega}_x$ are not defined in $\mathcal{S}(\mathfrak{A})$ for all $x \in \mathfrak{A}_L$, since for some x and y the functions $\omega_x^t(y)$ are almost periodic in $t \in \boldsymbol{R}$ (in the strong sense). Oscillations of the functions $\omega_x^t(y)$ are unimportant, as will be shown in Sec. IV., from the point of view of macroscopic quantities of the system (hence, also from the thermodynamical point of view). The locally fluctuating states ω_x^t converge, however, to *partial states* on certain smaller $\mathbf{C}^*$-algebras $\tilde{\mathfrak{A}} \subset \mathfrak{A}$. The partial description of the infinite system by such a subalgebra $\tilde{\mathfrak{A}} \subset \mathfrak{A}$ contains all the information concerning global (macroscopic) quantities of the system (we need not distinguish here between "observables at infinity" and "macroscopic observables", compare [2]). The limits $\bar{\omega}_x$ identified with the above mentioned partial states can be classified by values of some macroscopic observables. It will be proved that the initial perturbed vacuum state ω_x^0 is macroscopically different from all the $\bar{\omega}_x$ $(x \in \mathfrak{A}_L,\ \omega_x^0 \neq \omega^0)$. The stationary state ω^0 is unstable, in the just mentioned sense, with respect to all local perturbations.

II. BASIC PROPERTIES OF THE MODEL

The algebra of observables $\mathfrak{A}$ is the $\boldsymbol{C}^*$-inductive limit of the net of *local algebras* $\mathfrak{A}_X$, where the indices X are arbitrary finite subsets of the set $\boldsymbol{Z}$ of all integers with the partial ordering given by the set inclusion. $\mathfrak{A}_X$ is the $\mathbf{C}^*$-algebra generated by 1/2-spin creation and annihilation operators a_m^+, a_m $(m \in X)$ satisfying the commutation relations

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[1]) For definitions of technical terms see, e.g. [3] and [1].

$$[a_k, a_m] = [a_k^+, a_m] = 0 \text{ for } k \neq m,\ a_m^2 = 0, \tag{1}$$

$$a_m^+ a_m + a_m a_m^+ = 1 \qquad (k, m \in \mathbf{Z})$$

and a_m^+ is the adjoint of a_m.
Let the *vacuum state* be defined by

$$\omega^0 \in \mathcal{S}(\mathfrak{A}) : \omega^0(a_k^+ a_k) = 0 \text{ for all } k \in \mathbf{Z}. \tag{2}$$

This state describes the spin chain with all spins "pointing down" or, in the lattice gas interpretation, the lattice with all sites free of particles. If in some state ω there is $\omega(a_m^+ a_m) = 1$, we say that the m-th site in the lattice is occupied (equivalently, the m-th spin is pointing up). The simplicity of $\mathfrak{A}$ implies that the GNS-representation corresponding to the state ω^0 is faithful. It is easily seen that (2) defines unambiguously a pure state. Hence, the *vacuum representation* (i.e. the GNS-representation obtained from the vacuum state ω^0) is also irreducible. To simplify notations we shall identify the algebra $\mathfrak{A}$ with the range of its vacuum representation. Let $\mathcal{F}$ denotes the Hilbert space of this representation (called sometimes the *Fock space*) and let $\Omega^0 \in \mathcal{F}$ be the *vacuum vector* with the property

$$\omega^0(x) = (\Omega^0, x\Omega^0) \text{ for all } x \in \mathfrak{A}. \tag{3}$$

The symbol (Ω, Ω') denotes the scalar product of the vectors Ω and Ω' in $\mathcal{F}$, linear in the second variable Ω'. Clearly $\mathcal{F} = \mathfrak{A}\Omega^0$ due to the cyclicity of Ω^0 and the irreducibility of the representation. Define

$$h_m = a_m^+ a_m (a_{m+1}^+ + a_{m+1})\, a_{m+2} a_{m+2}^+, \tag{4}$$

$$H_{(j,k)} = \sum_{m=j+1}^{k-2} h_m. \tag{5}$$

The time evolution of our system is given in a usual manner [4] as a strongly continuous one-parameter group $\tau_t \in \operatorname{aut} \mathfrak{A}$ defined as the strong limit of a sequence of local automorphism groups $\tau_t^{(n)}$:

$$\tau_t^{(n)} x = \exp(\mathrm{i}tH_{(-n,n)})\, x \exp(-\mathrm{i}tH_{(-n,n)}) \tag{6n}$$

$$\tau_t x = \underset{n\to\infty}{\text{norm-lim}}\ \tau_t^{(n)} x \text{ for } x \in \mathfrak{A}_L. \tag{6}$$

Since $h_m \Omega^0 = 0$ for all $m \in \mathbf{Z}$, the state ω^0 is stationary and in the vacuum representation we can write

$$\tau_t x = \mathrm{e}^{\mathrm{i}tH} x\, \mathrm{e}^{-\mathrm{i}tH} \tag{6a}$$

and

$$H\Omega^0 = 0 \tag{6b}$$

for certain selfadjoint operator H in $\mathcal{F}$.

Let $\Omega_X \in \mathscr{F}$ be defined for all finite $X \subset \boldsymbol{Z}$ by

$$\Omega_X \equiv \left(\prod_{m\in X} a_m^+\right)\Omega^0. \tag{7}$$

The vectors Ω_X form an orthonormal basis in $\mathscr{F}$ (with $\Omega^0 = \Omega_\emptyset$, $\emptyset$ is the symbol for the empty set). Let $\mathscr{G}$ be the linear space of all finite complex linear combinations of vectors Ω_X. $\mathscr{G}$ is norm-dense in $\mathscr{F}$. Let H be a symmetric linear operator with the initial domain $\mathscr{G}$ formally given by

$$H = \sum_{m\in \boldsymbol{Z}} h_m. \tag{8}$$

Since $h_m \Omega_X = 0$ for $m \notin X$, the operator H is well defined on $\mathscr{G}$ and it is not difficult to see that $\mathscr{G}$ is H-invariant:

$$H\mathscr{G} \subset \mathscr{G}. \tag{9}$$

Each vector Ω_X can be written in the form

$$\Omega_x = \left(\prod_{j=1}^{r}\prod_{s=1}^{m_j} a_{k_j+s}^+\right)\Omega^0 = \Omega_r([k],[m]), \tag{10}$$

where $[k] \equiv [k]_r \equiv \{k_1, k_2, \ldots, k_r\}$ is an r-tuple of integers with $k_{j+1} > k_j + 1$ and $[m] \equiv [m]_r \equiv \{m_1, m_2, \ldots, m_r\} \subset \boldsymbol{Z}_+$ with $0 < m_j < k_{j+1} - k_j$ for $j = 1, 2, \ldots, r-1$, and $m_r > 0$. Let $\mathscr{G}_{[k]_r} \subset \mathscr{G}$ be the linear space spanned by all the vectors (10) with fixed $[k]_r$ and let $\mathscr{F}_{[k]_r}$ be the closure of $\mathscr{G}_{[k]_r}$. Then we have

$$\mathscr{F} = \bigoplus_{[k]_r} \mathscr{F}_{[k]_r}, \tag{11}$$

where the direct sum of Hilbert spaces is taken over all "permitted" r-tuples $[k]$ as specified above and over all nonnegative integers r. As a consequence of definitions we have

$$H\Omega_r([k],[m]) = \sum_{j=1}^{r} H_{(k_j,k_{j+1})}\Omega_r([k],[m]), \tag{12}$$

where $k_{r+1} \equiv +\infty$. The operators $H_{(k_j,k_{j+1})}$ occurring in (12) mutually commute. Similarly we get

$$H\mathscr{G}_{[k]_r} \subset \mathscr{G}_{[k]_r}. \tag{13}$$

Let $\mathscr{F}_N$ $(N \in \boldsymbol{Z}_+ \cup (+\infty))$ be the Hilbert space spanned by the vectors

$$\beta_m = a_1^+ a_2^+ \ldots a_m^+ \Omega^0 \tag{14}$$

with $m = 1, 2, \ldots, N$. Define a unitary mapping $V([k]_r)$ of $\mathscr{F}_{[k]_r}$ onto the tensor product

$$\bigotimes_{j=1}^{r} \mathscr{F}_{N_j}, \text{ where } N_j = k_{j+1} - k_j - 1, \quad . \ N_r = +\infty \tag{15}$$

by the relation

$$V([k]_r)\, \Omega_r([k], [m]) = \beta_{m_1} \otimes \beta_{m_2} \otimes \ldots \otimes \beta_{m_r}, \tag{16}$$

where $[m] = \{m_1, m_2, \ldots, m_r\}$. With the notation

$$H_N = \sum_{m=1}^{N-1} h_m \tag{17}$$

we can write

$$V([k]_r)\, H_{(k_j, k_{j+1})}\, \Omega_r([k], [m]) =$$

$$= \beta_{m_1} \otimes \ldots \otimes \beta_{m_{j-1}} \otimes (H_{N_j} \beta_{m_j}) \otimes \beta_{m_{j+1}} \otimes \ldots \otimes \beta_{m_r}, \tag{18}$$

and the action of H in $\mathscr{G}$ can be expressed by the actions of H_N in dense subspaces of $\mathscr{F}_N$ (or equivalently, of $\mathscr{F}_\infty$). The actions of H_N in $\mathscr{F}_\infty$ are given by the relations:

$$H_N \beta_1 = \beta_2 \tag{19a}$$

$$H_N \beta_m = \beta_{m-1} + \beta_{m+1}, \qquad m = 2, 3, \ldots, N-1, \tag{19b}$$

$$H_N \beta_N = \beta_{N-1} \tag{19c}$$

$$H_N \beta_k = 0 \quad \text{for} \quad k > N. \tag{19d}$$

The relations (19a, b) are valid also for $N = +\infty$, in which case we can write H instead of H_∞. Now we are ready to prove

II. 1. Proposition. *The operator H defined by* (8) *on the domain $\mathscr{G}$ is an essentially selfadjoint operator, the closure of which is identical with the generator H of τ_t occurring in* (6).

Proof. A direct sum of bounded essentially selfadjoint operators is an essentially selfadjoint operator. For $N \in \mathbf{Z}_+ \cup (+\infty)$ and

$$\beta = \sum_{m=1}^{K} c_m \beta_m (c_m \in \mathbf{C}, K < +\infty)$$

we have from (19)

$$\|H_N \beta\| \leq 2 \|\beta\| \tag{20a}$$

and (20a) combined with (12) and (18) implies

$$\|H\Omega\| \leq 2r \|\Omega\| \text{ for all } \Omega \in \mathscr{G}_{[k]_r}. \tag{20b}$$

Relation (20b) together with (13) and (11) proves the essential selfadjointness of H. To prove the convergence of (6n) for $n \to +\infty$ to the relation (6a) with H from

(8) it suffices to prove $\lim_{n\to+\infty} \|[\exp(itH) - \exp(itH_{(-n,n)})]\,\Omega\| = 0$ for all $\Omega \in \mathcal{G}$ and this is a consequence of (20b), (13), (11) and of the fact that $H_{(-n,n)}$ converges strongly to H if restricted to the subspace $\mathcal{F}_{|k|_r}$, q.e.d.

The unique selfadjoint extension of H from (8) is denoted by the same symbol H. The explicit expression for the unitary group $\exp(-itH)$ is derived in the next section.

III. THE GREEN FUNCTION

The matrix elements

$$(\Omega_X, e^{-itH}\Omega_Y) \text{ for all finite } X, Y \subset \mathbf{Z} \tag{21}$$

determine the automorphism group of our system. According to (12), (13), (16) and (18) we have

$$\begin{aligned}(\Omega_r([k],[m]), e^{-itH}\Omega_{r'}([k'],[m'])) = \\ = \delta_{rr'}\delta_{|k||k'|}(\beta_{m_r}, e^{-itH}\beta_{m_r'})\prod_{j=1}^{r-1}(\beta_{m_j}, e^{-itH_{N_j}}\beta_{m_j'}).\end{aligned} \tag{22}$$

Each matrix element (21) is of the form (22) and we have to calculate

$$(\beta_m, e^{-itH_N}\beta_n) \text{ for } m, n = 1, 2, \dots, N; \quad N \in \mathbf{Z}_+. \tag{23}$$

and also the matrix elements

$$(\beta_m, e^{-itH}\beta_n) \text{ for all } m, n \in \mathbf{Z}_+, \tag{24}$$

which coincide with (23) for $N = +\infty$. To get the explicit form of (23) we shall solve the eigenvalue problem for H_N in the N-dimensional Hilbert space $\mathcal{F}_N$. The action of H_N in $\mathcal{F}_N$ is given by (19a—c). Writing the eigenvector α_E corresponding to an eigenvalue E of H_N in the form

$$\alpha_E = \sum_{m=1}^{N} c_m(E)\,\beta_m \tag{25}$$

we get the eigenvalue equation in the form

$$Ec_1(E) = c_2(E) \tag{26a}$$

$$Ec_m(E) = c_{m-1}(E) + c_{m+1}(E) \text{ for } m = 2, 3, \dots, N-1 \tag{26b}$$

$$Ec_N(E) = c_{N-1}(E).$$

Equations (26) give us the expressions

$$c_m(E) = \mathrm{U}_{m-1}(E/2)\,c_1(E), \tag{27}$$

where $\mathrm{U}_n(z)$ are n-th order polynomials satisfying the recurrent relations

$$\mathrm{U}_{n+1}(z)=2z\mathrm{U}_n(z)-\mathrm{U}_{n-1}(z), \quad \mathrm{U}_0(z)=1, \quad \mathrm{U}_1(z)=2z. \tag{28}$$

The unique solution of (28) is the sequence of the Tshebyshev polynomials of the second kind:

$$\mathrm{U}_{m-1}(z)=\frac{\sin(m \arccos z)}{\sin(\arccos z)}. \tag{29}$$

Using (27) and (28), equation (26c) is written in the form

$$\mathrm{U}_N(E/2)=0, \tag{30}$$

which si in fact the secular equation of the system (26). The expression (29) shows that the solutions of (30) are

$$E=E_j=2\cos\left(\frac{j\pi}{N+1}\right), \qquad j=1,2,\ldots,N. \tag{31}$$

Substitution of (31) into (27) and normalization of the eigenfunctions $\alpha_j=\alpha_{E_j}$ gives

$$c_{jm}=c_m(E_j)=[2/(N+1)]^{1/2}\sin[jm\pi/(N+1)]. \tag{32}$$

Let us introduce the almost periodic functions of $z\in\boldsymbol{R}$ by

$$J_m^{(N)}(z)=\frac{\mathrm{i}^m}{N+1}\sum_{j=1}^{N}\exp\left[-\mathrm{i}z\cos\left(\frac{j\pi}{N+1}\right)\right]\cos\left(m\frac{j\pi}{N+1}\right), \tag{33}$$

which are integral sums for the Bessel functions $J_m(z)$ in their Sommerfeld integral representation:

$$J_m(z)=\frac{\mathrm{i}^m}{\pi}\int_0^{\pi}\mathrm{e}^{-\mathrm{i}z\cos u}\cos(mu)\,\mathrm{d}u. \tag{34}$$

Since $c_{jm}=(\beta_m,\alpha_j)$, the results (31) and (32) lead to the following statement:

III. 1. Proposition. *The matrix elements* (23) *can be expressed in terms of* (33) *by the formula*

$$(\beta_m,\mathrm{e}^{-\mathrm{i}tH_N}\beta_n)=(-\mathrm{i})^{m-n}J_{m-n}^{(N)}(2t)-(-\mathrm{i})^{m+n}J_{m+n}^{(N)}(2t). \tag{35}$$

Proof. We obtain (35) by straightforward calculations using the completeness of the orthonormal system $\{\alpha_j\}$ in $\mathcal{F}_N$, q.e.d.

Expression for the matrix elements (24) might be obtained either by taking the limit $N\to+\infty$ in (35) or by another method without using the results for finite subchains. The use of (35) gives the desired result immediately, since

$$\lim_{N\to+\infty}J_m^{(N)}(z)=J_m(z) \tag{36}$$

and the strong operator limit in $\mathcal{F}_\infty$

$$\text{s}-\lim_{N\to+\infty} e^{itH_N}=e^{itH}. \tag{37}$$

Hence, we have proved

III. 2. Proposition. *The matrix elements* (24) *are given by*

$$(\beta_m, e^{-itH}\beta_n)=(-i)^{m-n}J_{m-n}(2t)-(-i)^{m+n}J_{m+n}(2t), \tag{38}$$

where $J_m(z)$ *are the Bessel functions.*

The propositions III.1. and III. 2. together with (22) give the explicit form of all the matrix elements (21) and this expresses the Green function describing the time evolution in our model.

III. 3. Note. The formula (38) can be obtained directly from the known action of H in $\mathscr{F}_\infty$ given in (19). The relations (19a, b) with $N=+\infty$ lead to the recurrent relations for the matrix elements $(\beta_n, H^k\beta_m)$ with the solution

$$(\beta_m, H^k\beta_n)=\binom{k}{\frac{k+n-m}{2}}-\binom{k}{\frac{k-n-m}{2}}. \tag{39}$$

The combinatorial numbers $\binom{k}{x}\neq 0$ only for nonnegative integers x. The power expansions of $\exp(itH)$ and (39) give (38).

III. 4. Proposition. *The spectrum of H consists of a simple eigenvalue* 0 *and an absolutely continuous part.*

Proof. It suffices to prove that the measures m_Ω on the real line R,

$$m_\Omega(dz)\equiv(\Omega, P_H(dz)\,\Omega) \tag{40}$$

are absolutely continuous (a.c.) with respect to the Lebesgue measure for all Ω: $(\Omega, \Omega^0)=0$, $\Omega\in\mathscr{F}$. In (40) P_H denotes the spectral measure of H. The vectors Ωin $\mathscr{F}$ with m_Ω a.c. form an H-invariant closed subspace in $\mathscr{F}$ and we have to prove the a.c. of m_Ω for all the $\Omega=\Omega_X(X\neq\emptyset)$ only. The matrix element (21) with $Y=X$ is a Fourier transform of such a measure. According to (22) and [5] (Proposition IV. 1.) it suffices to prove that

$$(\beta_m, e^{-itH}\beta_m)=0(t^{-\delta}) \text{ for } t\to\infty \text{ with } \delta>1 \tag{41}$$

for all $m>0$. According to (38) and the recurrent relations for the Bessel functions

$$J_{p+1}(z)+J_{p-1}(z)=2p/zJ_p(z) \tag{42}$$

we have

$$(\beta_m, e^{-itH}\beta_m)=\frac{1}{t}\sum_{j=1}^{m}(-1)^{j+1}(2j-1)\,J_{2j-1}(2t). \tag{43}$$

A finite linear combination of the Bessel functions behaves for large t as $O(t^{-1/2})$ and (43) implies (41) with $\delta=3/2$, q.e.d.

IV. INSTABILITY OF THE VACUUM

We are interested here in the time evolution of local perturbations of the state ω^0 for large times, i.e. we shall investigate the expressions

$$\omega_x^t(y) \equiv \omega^0(x^*(\tau_t y)x), \quad x \in \mathfrak{A}_L, \ \omega^0(x^*x)=1, \quad y \in \mathfrak{A} \tag{46}$$

for $t \to \infty$. Limits of expressions (46) for $t \to \infty$ do not exist for all the $y \in \mathfrak{A}$ and an arbitrary $x \in \mathfrak{A}_L$.

IV. 1. An example. Let $x \equiv a_1^+ a_{N+2}^+$, $y \equiv a_N^+ a_N$ with $N>2$.
According to (22) we can write

$$\omega_x^t(y) = (a_1^+ a_{N+2}^+ \Omega^0, \mathrm{e}^{itH} a_N^+ a_N \, \mathrm{e}^{-itH} a_1^+ a_{N+2}^+ \Omega^0) = |(\beta_N, \mathrm{e}^{-itH_N} \beta_1)|^2,$$

which is an almost periodic function of t (in the strong sense).

Take $x \in \mathfrak{A}_Y$ in (46) and let $[Y] \subset \mathbf{Z}$ be the minimal set of the form $\{l_1, l_1+1, l_1+2, \ldots, l_2-1, l_2\}$ containing Y. Denote by $\mathfrak{A}_{[Y]}$ the $\mathbf{C}^*$-subalgebra of $\mathfrak{A}$ generated by all the $\mathfrak{A}_X$ with $X \cap [Y] = \emptyset$. Every element y of $\mathfrak{A}_{[Y]}$ is a norm-limit of local elements from $\mathfrak{A}_{[Y]} \cap \mathfrak{A}_L$. Each local element can be written in the form of a finite linear combination of identify and of monomials

$$\left(\prod_{j=1}^{K_1} a_{m_j}^+\right)\left(\prod_{i=1}^{K_2} a_{s_i}^+ a_{s_i}\right)\left(\prod_{k=1}^{K_3} a_{n_k}\right), \tag{47}$$

where all the $K_1+K_2+K_3$ indices m_j, s_i, n_k are mutually different. Let $\mathscr{S}(\mathfrak{A}_{[Y]})$ be a set of all states on $\mathfrak{A}_{[Y]}$. Define $\bar{\omega}_{[Y]} \in \mathscr{S}(\mathfrak{A}_{[Y]})$ by

$$\bar{\omega}_{[Y]}(a_j^+ a_j) = 0 \text{ for } j < l_1(Y) \equiv \min\{m \mid m \in Y\}, \tag{48a}$$

$$\bar{\omega}_{[Y]}(a_j^+ a_j) = 1 \text{ for } j > l_2(Y) \equiv \max\{m \mid m \in Y\}. \tag{48b}$$

The definition (48) determines a unique pure state on the $\mathbf{C}^*$-algebra $\mathfrak{A}_{[Y]}$ (compare with the definition (2) of ω^0).

IV. 2. Proposition. *Restrictions of the states ω_x^t ($x \in \mathfrak{A}_Y$) to the algebra $\mathfrak{A}_{[Y]}$ converge for $t \to \infty$ in the w^*-topology of $\mathfrak{A}_{[Y]}^*$ to the state $\bar{\omega}_x \in \mathscr{S}(\mathfrak{A}_{[Y]}) : \bar{\omega} \equiv w\omega^0 + (1-w)\,\bar{\omega}_{[Y]}$, where $w \equiv |\omega^0(x)|^2$ and the symbol ω^0 denotes here the vacuum state on $\mathfrak{A}_{[Y]}$.*

Proof. Let us write

$$x\Omega^0 = c_1 \Omega^0 + c_2 \Omega^x \text{ with } (\Omega^x, \Omega^0) = 0, \ \|\Omega^x\| = 1. \tag{49}$$

Clearly $c_1 = \omega^0(x)$ and

$$|c_1|^2 = w, \quad |c_2|^2 = 1 - w. \tag{50}$$

We get for (46) with the help of (49)

$$\omega_x^t(y) = w\omega^0(y) + (1-w)\,(\Omega^x,\ \tau_t y\Omega^x) + c_1^* c_2(\Omega^0,\ y\ \mathrm{e}^{itH}\Omega^x) + \\ + c_1 c_2^*(\Omega^x,\ \mathrm{e}^{itH} y\Omega^0). \tag{51}$$

According to III. 4. and (49) the two matrix elements of the operators $\exp(\pm itH)$ occurring in (51) converge to zero as t approaches infinity. Hence, it suffices to prove that

$$\lim_{t\to\infty}(\Omega^x,\ \tau_t y\Omega^x) = \bar{\omega}_{[Y]}(y) \text{ for all } y\in\mathfrak{A}_{[Y]}\cap\mathfrak{A}_L \tag{52}$$

and for an arbitrary $x\in\mathfrak{A}_Y$. Since the definition (48) is unambiguous it suffices to prove the *existence* of all the limits in (52) and to prove the *equality* (52) for $y = a_j^+ a_j$ ($j\in\boldsymbol{Z}\setminus[Y]$) only. In proving (52) we can take, moreover, only those y in (52) which are monomials of the form (47). Since

$$\Omega_x = \sum_{X\subset Y} c(X)\,\Omega_X \text{ with } c(X)\in\boldsymbol{C} \text{ and } c(\emptyset) = 0, \tag{53}$$

we can write the matrix element $(\Omega^x,\ \tau_t y\Omega^x)$ in (52) as a finite linear combination of the matrix elements

$$(\Omega_{X'},\ \mathrm{e}^{itH} y\ \mathrm{e}^{-itH}\Omega_X) = \\ = \sum_{[m'']} [(\Omega_{r'}([k'],[m'],\ \mathrm{e}^{itH} y\Omega_r([k],[m''])) \,. \\ .\,(\Omega_r([k],[m'']),\ \mathrm{e}^{-itH}\Omega_r([k],[m]))], \tag{54}$$

where we used the identification according to (10) and where

$$\min(k_1', k_1) \geqslant l_1(Y) - 1,\ \max(k'_{r'} + m'_{r'},\ k_r + m_r) \leqslant l_2(Y). \tag{55}$$

The elements (54) can be nonvanishing (with $X', X\subset Y$) only for such monomials y of the form (47) ($y\in\mathfrak{A}_{[Y]}$, $y\neq 1$) which have all the indices m_j, s_i, $n_k > l_2(Y)$ and if there is an index $[m'']$ in the sum in (54) such that

$$y\Omega_r([k],[m'']) = \Omega_r([k],[m''']) \tag{56}$$

for some $[m''']$. Due to (55) and the choice of y the r-tuples $[m'']$ and $[m''']$ differ one from another at most in the r-th component m_r''. For another choice of a monomial y all the matrix elements (54) vanish and the limit in (52) exists. For monomials y satisfying (56) the matrix elements (54) have the form

$$\delta_{r'r}\delta[k'][k] \sum_{m=1}^{\infty} (\beta_{m_r'},\ \mathrm{e}^{itH} y\beta_m)\,(\beta_m,\ \mathrm{e}^{-itH}\beta_{m_r}) \prod_{j=1}^{r-1} \delta_{m_j' m_j}\,. \tag{57}$$

If there is $K_1 + K_3 \neq 0$ in the expression (47) for the monomial y, then the sum in (57) is finite and according to (38) the limit of (57) for $t\to\infty$ vanishes. A nonzero limit of (57) might be obtained only for monomials $y\in\mathfrak{A}_{[Y]}$ of the type

$$y=\prod_{i=1}^{K} a^+_{s_i} a_{s_i} \text{ (let } s_1<s_2<\ldots<s_K). \tag{58}$$

For the $s_1<l_1(Y)$ elements (54) vanishes. In other cases (i.e. $s_1>l_2(Y)$) a substitution of (58) into (57) and a use of completness of $\{\beta_m\}$ lead to the following form of (54):

$$\delta_{r'r}\delta_{|k'||k|}\,[\delta_{m'_r m_r}-\sum_{m=1}^{s_K-1}(\beta_{m'_r},\, e^{itH}\beta_m)\,.$$

$$.\,(\beta_m\,,\, e^{-itH}\beta_{m_r})]\prod_{j=1}^{r-1}\delta_{m'_j m_j}\,. \tag{59}$$

The finite sum of (59) converges to zero and we have

$$\lim_{t\to\infty}(\Omega_{X'},\, e^{itH} y\, e^{-itH}\Omega_X)=\delta_{X'X}\,, \tag{60}$$

which gives together with (49) and (53) the desired equality (52), q.e.d.

Let us define a sequence of local observables in $\mathfrak{A}$ by

$$\gamma_n=\frac{1}{2n+1}\sum_{j=-n}^{n} a^+_j a_j\,, \qquad n=0,1,2,\ldots \tag{61}$$

Since ω^0 and $\bar{\omega}_{|Y|}\in\mathcal{S}(\mathfrak{A}_{|Y|})$ are pure (hence primary) states and

$$\lim_{n\to\infty}\omega^0(\gamma_n)=0, \quad \lim_{n\to\infty}\bar{\omega}_{|Y|}(\gamma_n)=1/2, \tag{62}$$

an application of a simple lemma from [2] (Lemma 6) gives disjointness of the states ω^0 and $\bar{\omega}_{|Y|}$ for all finite $Y\subset \boldsymbol{Z}$. The sequence γ_n in (61) is an example of a "generalized observable" in the sense that the limit

$$w-\lim_{n\to\infty}\pi(\gamma_n)\equiv\gamma_\pi\in[\pi(\mathfrak{A})]'' \tag{63}$$

exists in some nonzero representation π of the quasilocal algebra $\mathfrak{A}$. Specifically, the weak-operator limits (63) exist in GNS-representations induced by both ω^0 and $\bar{\omega}_{|Y|}$ and γ_π is a macroscopic quantity in these representations (compare [2]). According to (62) the states ω^0 and $\bar{\omega}_{|Y|}$ are macroscopically different. These considerations and the proposition IV. 2. show

IV. 3. Corollary. *Every local perturbation of the vacuum state ω^0 evolves spontaneously into a state macroscopically different from ω^0.*

This expresses the instability of ω^0. Another unstable state of our system is the state with all the sites in the lattice occupied (or, with all the spins pointing up).

REFERENCES

[1] Sakai S., *C*-Algebras and W*-Algebras*. Springer, Berlin-Heidelberg-New York 1971.
[2] Hepp K., Helv. Phys. Acta *45* (1972), 237.
[3] Hugenholtz N. M., *States and Representations in Statistical Mechanics*. In: *Mathematics of Contemporary Physics* (Ed. R. F. Streater), Academic Press, London-New York 1972.
[4] Robinson D. W., Commun. Math. Phys *7* (1968), 337.
[5] Bóna P., Acta Phys. Slov. *25* (1975), 3.

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