TIME EVOLUTION AUTOMORPHISMS IN GENERALIZED MEAN-FIELD THEORIES*)

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A general class of time evolutions τ^Q of infinite quantum systems is rigorously defined. It generalizes thermodynamic limits of polynomial mean-field evolution of quantum spin lattices, the simplest case of which is the strong coupling version of the quasi spin B.C.S.-model of superconductivity. A distinguished feature of the considered type of time evolution is the τ^Q-non-invariance of the usually considered C^*-algebra $\boldsymbol{A}$ of quasilocal observables of the infinite system. A larger C^*-algebra $\boldsymbol{C}$ containing $\boldsymbol{A}$ as a subalgebra is introduced in such a way that τ^Q has a natural extension to a one parameter group *-automorphisms of $\boldsymbol{C}$. The algebra $\boldsymbol{C}$ contains a commutative subalgebra of classical observables (consisting of the intensive observables of the large quantal system determined by a Lie group G action $\sigma(G) \in$ *-aut $\boldsymbol{A}$) denoted by $\boldsymbol{N}$ which is τ^Q invariant and the restriction of τ^Q to $\boldsymbol{N}$ reproduces the classical Hamiltonian flow φ^Q corresponding to the chosen classical Hamiltonian function Q on the classical phase space of the intensive observables. The evolution τ^Q is determined uniquely by the classical Hamiltonian function Q as well as by the action $\sigma(G)$. Continuity properties of τ^Q are considered and reviewed.

1. INTRODUCTION

The mean-field approximations to models in statistical mechanics were introduced at the beginning of our century to obtain a certain understanding of the mechanism of phase transitions, cf. [1] and [2] for reviews. The approximation consists, roughly speaking, of replacing of a considered system of interacting 'individuals' by a system of independent 'individuals' moving in an appropriately chosen effective external field. There is a class of Hamiltonian operators introduced in [3] which are invariant with respect to all permutations of 'individuals' in the system and with k-body interaction constants proportional to N^{-k+1}, where N is the number of the 'individuals' collected in the whole system, such that the systems endowed with such Hamiltonian generators of the dynamics behave in the thermodynamic limit $N \to \infty$ analogously to the above mentioned picture of the motion in an effective 'mean-field'. We shall show how the rigorous thermodynamic limit of such 'mean-field Hamiltonians' (or of the corresponding time evolution of the whole collection of infinite number of 'individuals') looks like. Let us stress that the 'mean-field theory' obtained as the thermodynamic limit of the dynamics corresponding to a 'mean-field Hamiltonian' from [3] corresponding exactly to the given (N-dependent) interaction between the 'individuals', and it is not, in this sense, any approximation. The simplest model of this type occurring in physical theories is, perhaps, the strong coupling version

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of the quasi-spin formulation of the B.C.S.-model of superconductivity, as it was described in [4]. We shall describe here a general class of time evolutions of infinite quantal systems including the last mentioned type of systems. It will be seen that, contrary to certain conclusions in some classical papers (see, e.g., [4], [5] and [6]), no 'pathologies' occur in the algebraic description of these models. One of the most important results of our analysis is that the time evolution in these mean-field theories cannot be described as a one-parameter group of C^*-automorphisms of the usually introduced algebra $\boldsymbol{A}$ of quasilocal observables, but a larger C^*-algebra $\boldsymbol{C}$ can be defined in a natural way such that the mean-field theory determines an automorphic dynamics of $\boldsymbol{C}$. Our approach generalizes and makes more compact and more rigorous some of previous attempts at defining the mean-field time evolution in certain GNS-representation of $\boldsymbol{A}$ corresponding to some equilibrium (hence time-invariant) states, cf. [8]. We shall give an explicit formula for the time evolution of an arbitrary element $x \in \boldsymbol{C}$, hence also for any quasilocal element $x \in \boldsymbol{A}$. Such a formula was given in [3] for classical (intensive) quantities of the large system only; these quantities are also included as central element of the algebra $\boldsymbol{C}$. Also equilibrium (KMS-) states, ground states and their decompositions (hence also corresponding symmetry breakdown) are analyzable in a straightforward way in the framework of our formalism.

We shall first describe the general theory of the considered class of mean-field theories and only after that we shall sketch the proof of the fact that the mean-field theories from [3] are included in our scheme. The resulting picture of the behaviour of the infinite quantum-mechanical system is very intuitive even in the general case: Any individual subsystem moves 'independently of its neighbours' in a time-dependent external field according to Schrödinger equation with a time-dependent Hamiltonian (depending on a given initial state of the infinite system). The 'external field' satisfies classical Hamiltonian equations of motion. Any change of initial states of any finite number of individuals does not affect the 'external' field. The values of this classical field are, however, equal to the values of corresponding intensive quantities of the large system, hence are equal to the (quantal as well as statistical) expectation values of certain distinguished observables of the individual ('small') subsystems. No external field is put by hand into these models, and the resulting 'mean-field' described by classical equations of motion is an exact consequence of the internal structure of the interactions in the infinite collection of quantal 'individual' subsystems. By an appropriate change of initial conditions of an infinite subset of 'individuals' the values of the field can be changed, and the further evolution of the whole system depends on properties of the corresponding classical system of intensive quantities: if this classical system of the 'mean-fields' is unstable in a neighbourhood of the chosen individual value of the intensive quantities, the above mentioned (arbitrarily small) change of initial states of an infinite subset of individuals can result in a qualitative difference in the behaviour of the whole system. This brings in mind a possibility to use our mean-field theories for modelling certain social phenomena.

Any mean-field theory of the considered class is determined in our approach by a connected Lie group G, by its strongly continuous unitary representation $U(G)$ in a separable Hilbert space H and by an infinite (countable) set Π which will be identified here with the set Z_+ of non-negative integers. The dynamics of the theory is specified by a classical Hamiltonian function $Q \in C^\infty(\mathcal{g}^*, \mathbb{R})$ on the dual space $\mathcal{g}^*$ to the Lie algebra $\mathcal{g}$ of the group G. These objects determine the kinematics as well as the dynamics of the whole infinite system. For notational convenience, we shall proceed in the following order: In sec. 2, the classical system of the 'mean-fields' on the generalized (finite dimensional) phase space $\mathcal{g}^*$ is defined, the 'large' quantal system consisting of infinitely many quantal 'individuals' is describe in the sec. 3 and the time evolution τ^Q is determined in sec.4. The last sec. 5 is devoted to a description of the relation of the presented theory to the 'conventional cases' in which the resulting theory is given by a thermodynamic limit of local Hamiltonian evolutions (in these cases the function Q is a polynomial and the Hilbert space H is finite dimensional, the group G being compact).

2. THE CLASSICAL SUBSYSTEM

Let G be the given connected Lie group and let $\mathcal{g}$ be its Lie algebra. Let $[\beta, \chi] \in \mathcal{g}$ be the Lie-bracket of elements β and χ in $\mathcal{g}$. Let $\mathcal{g}^*$ be the dual of $\mathcal{g}$ considered as a finite dimensional differentiable manifold with the topology of the linear vector space. For any point $F \in \mathcal{g}^*$, let the tangent space $T_F\mathcal{g}^*$ of $\mathcal{g}^*$ in F be identified in the canonical way with the linear space $\mathcal{g}^*$ itself, and let its dual $T^*_F\mathcal{g}^*$ be identified with the Lie algebra $\mathcal{g} = \mathcal{g}^{**}$. Let $f, g \in C^\infty(\mathcal{g}^*, \mathbb{R})$ be arbitrary infinitely differentiable real-valued functions on $\mathcal{g}^*$, and let $\mathrm{d}_F f \in T^*_F\mathcal{g}^* = \mathcal{g}$ be the value at F of the exterior differential $\mathrm{d}f$ of f. A Poisson structure on $\mathcal{g}^*$ is determined by the two-contravariant antisymmetric tensor field λ:

$$(2.1) \qquad \lambda_F(\mathrm{d}_F f, \mathrm{d}_F g) := -F([\mathrm{d}_F f, d_F g]), \quad F \in \mathcal{g}^*, \quad f, g \in C^\infty(\mathcal{g}^*).$$

Here $F(\beta)$ denotes the value of the linear functional $F \in \mathcal{g}^*$ at the point $\beta \in \mathcal{g}$; in (2.1) we have $\beta := [\mathrm{d}_F f, \mathrm{d}_F g]$. A unique Hamiltonian vector field σ_f on $\mathcal{g}^*$ corresponds to any $f \in C^\infty(\mathcal{g}^*, \mathbb{R})$; it is determined by

$$(2.2) \qquad \mathrm{d}g(\sigma_f) := [f, g] := \lambda(\mathrm{d}f, \mathrm{d}g), \quad g \in C^\infty(\mathcal{g}^*, \mathbb{R}).$$

We have defined here the Poisson bracket $[f, g]$ of two differentiable functions f and g, too. This Poisson bracket satisfies all the properties of the Poisson brackets on simplectic manifolds except of the nondegeneracе: If $[f, g] = 0$ for all g, then f need not be constant in the case of Poisson manifolds, although it is necessarily constant on a simplectic manifold with the usual definition of $[\cdot, \cdot]$, cf. [9]. The vector field σ_f is complete if it generates the flow φ^f, i.e. φ^f is a one-parameter group of diffeomorphisms of $\mathcal{g}^*$ the differential of which is given by σ_f. Any diffeomorphism

φ of $\mathscr{g}^*$ conserving the Poisson brackets in the sense that

$$[\varphi^*f, \varphi^*g] = \varphi^*[f, g], \tag{2.3}$$

(φ^* is the pull-back of φ) is called a Poisson morphism (here it is even an automorphism). Any flow of the form φ^f consists of a one-parameter group $t \mapsto \varphi_t^f$ of Poisson automorphisms. Denote by $f_\beta \in C^\infty(\mathscr{g}^*, R)$ ($\beta \in \mathscr{g}$) the function $f_\beta(F) := F(\beta)$ for all $F \in \mathscr{g}^*$. Then

$$[f_\beta, f_\chi](F) = -F([\beta, \chi]) = -f_{[\beta,\chi]}(F). \tag{2.4}$$

Let G act on $\mathscr{g}$ by the canonically defined adjoint action $Ad(g)$ ($g \in G$), and let Ad^* be the coadjoint representation of G on $\mathscr{g}^*$:

$$Ad^*(g)\,F(\beta) := F(Ad(g^{-1})\,\beta), \quad \beta \in \mathscr{g}, \quad F \in \mathscr{g}^*, \quad g \in G. \tag{2.5}$$

Any operator $Ad^*(g)$ is a Poisson morphism. The subset of $\mathscr{g}^*$ given by

$$O_F := [Ad^*(g)\,F: g \in G] =: \text{an } Ad^*\text{-orbit of } G \tag{2.7}$$

is invariant with respect to any flow of Poisson automorphisms, cf. [10], for any given $F \in \mathscr{g}^*$. Let $Q \in C^\infty(\mathscr{g}^*, R)$ be a fixed function with the corresponding complete Hamiltonian vector field o_Q and the flow φ^Q. Then there is a unique function g_Q: $R \times \mathscr{g}^* \to G$ satisfying the relations:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g_Q(t, F) = \mathrm{d}_F Q \in \mathscr{g}, \quad g_Q(0, F) = e \tag{2.8}$$

(the identity of G),

$$g_Q(s, \varphi_t^Q(F))\, g_Q(t, F) = g_Q(s + t, F), \quad F \in \mathscr{g}^*, \quad t, s \in R. \tag{2.9}$$

The cocycle g_Q generates the flow φ^Q in the sense that we have:

$$\varphi_t^Q(F) = Ad^*(g_Q(t, F))\,F, \quad t \in R, \quad F \in \mathscr{g}^*. \tag{2.10}$$

This formula will enable us to determine the time evolution τ^Q of the infinite quantal system with a help of the classical time evolution φ^Q on the (generalized) phase space $\mathscr{g}^*$ of the 'classical fields'.

3. THE LARGE QUANTAL SYSTEM

Let H_p and $U_p(G)$ ($p \in Z_+$) be copies of a complex separable Hilbert space H and a weakly continuous unitary representation $U(G)$ in H which are identified by unitary operators $u_p: H \to H_p$, $U_p(g) = u_p\,U(g)\,u_p^{-1} =: \pi_p(U(g))$. We define similarly $\pi_p(A) := u_p A u_p^{-1}$ for other linear operators A on H. Let J be any finite subset of Z_+ and let

$$A^J := \bigotimes_{p\in J} \pi_p(L(H)) \tag{3.1}$$

be the W^*-tensor product of $|J| :=$ card J copies of the von Neumann algebras $L(H)$ of all bounded operators on H. The von Neumann algebras A^J form a net w.r.t. the naturally defined inclusions. The C^*-inductive limit of this net is the usually considered C^*-algebra of quasilocal observables A, which can be defined also as a C^*-tensor product of the infinite set of C^*-algebras, [11]:

$$A := \bigotimes_{J p \in Z_+} \pi_p(L(H)) . \tag{3.2}$$

The linear combinations of elements $x \in A$ of the form

$$x := \bigotimes_{p \in J} \pi_p(x_p) , \quad x_p \in L(H) , \quad J \subset Z_+ \tag{3.3}$$

(J finite), form a dense subset of A. Let $\sigma(g) \in$ *-aut $A(g \in G)$ be determined by

$$\sigma(g)(x) := \bigotimes_{p \in J} \pi_p(U(g) x_p U(g^{-1})) , \quad g \in G , \tag{3.4}$$

for all x of the form (3.3). We shall denote by the same symbol $\sigma(g)$ the unique extension of the automorphism from (3.4) to a C^*-automorphism of the double dual A^{**} of A. We shall identify A with its universal representation π_u in the Hilbert space H_u:

$$\pi_u(x) := \bigoplus_{\omega \in S(A)} \pi_\omega(x) , \quad H_u := \bigoplus_{\omega \in S(A)} H_\omega , \tag{3.5}$$

where $(\pi_\omega, H_\omega, \Omega_\omega)$ is the canonical cyclic (GNS) representation in the Hilbert space H_ω with the cyclic vector Ω_ω corresponding to any state ω from the set $S(A)$ of all states on the algebra A. The von Neumann algebra A^{**} will be identified with the weak-operator closure (the bicommutant) $\pi_u(A)''$ of $\pi_u(A)$.

Let X_β $(\beta \in \mathcal{g})$ be selfadjoint generators of $U(G)$:

$$U(\exp(t\beta)) =: \exp(-\mathrm{i}tX_\beta) , \quad t \in \mathbb{R} , \quad \beta \in \mathcal{g} . \tag{3.6}$$

Let

$$X_{\beta J} := \frac{1}{|J|} \sum_{p \in J} \pi_p(X_\beta) , \tag{3.7}$$

hence $\exp(\mathrm{i}tX_{\beta J}) \in A^J \subset A$. Let Z be the center of A^{**}, $Z := A'' \cap A'$, and let $p_G \in Z$ be the supremum of all projections $p \in Z$ satisfying the following three conditions:

(3.8i) The unitary groups $t \mapsto \exp(\mathrm{i}tX_{\beta J})\, p$ on the Hilbert space pH_u are strongly continuous for all finite $J \subset Z_+$ and for all $\beta \in \mathcal{g}$.

(3.8ii) The limits $\exp(\mathrm{i}tX_{\beta\Pi})\, p := \sigma - \lim_J \exp(\mathrm{i}tX_{\beta J})\, p$ $(\beta \in \mathcal{g},\ t \in \mathbb{R})$ exist in the $\sigma(A^{**}, A^*)$-topology.

(3.8iii) The groups $t \mapsto \exp(\mathrm{i}tX_{\beta\Pi}) := \exp(\mathrm{i}tX_{\beta\Pi})\, p_G$ $(\beta \in \mathcal{g})$ are strongly continuous in $L(H_u)$.

Then the mapping

$$\beta \mapsto \exp(\mathrm{i}X_{\beta\Pi}) \in p_G Z \tag{3.9}$$

is a strongly continuous unitary representation of the abelian group $\mathscr{g}$ (the group operation is here the vector addition) in the Hilbert space $p_G H_u$. According to the SNAG theorem, there is a unique projection valued measure $E_\mathscr{g}$ on $\mathscr{g}^*$ (which can be identified with the dual group $\hat{\mathscr{g}}$ by association with any $F \in \mathscr{g}^*$ the character $\chi_F: \beta \mapsto \chi_F(\beta) := \exp(\mathrm{i}F(\beta)))$ such that

$$\omega(x \exp(\mathrm{i}X_{\beta\Pi})\, y) = \int_{\mathscr{g}^*} \exp(\mathrm{i}F(\beta))\, \omega(xE_\mathscr{g}(dF)\, y) \tag{3.10}$$

for all $x, y \in A^{**}$, $\beta \in \mathscr{g}$ and all normal states ω on A^{**} such that $\omega(p_G) = 1$. The projection valued measure $E_\mathscr{g}$ is G-equivariant in the sense:

$$\sigma(g)\,(E_\mathscr{g}(B)) = E_\mathscr{g}(Ad^*(g)\, B)\,, \quad g \in G\,, \quad \text{Borel}\, B \subset \mathscr{g}^*\,. \tag{3.11}$$

Specifically, we have $\sigma(G)$-invariance of p_G:

$$\sigma(g)\,(p_G) = p_G = E_\mathscr{g}(\mathscr{g}^*)\,. \tag{3.12}$$

We shall work mostly with the states $\omega \in S(A)$ (resp. with the normal states $\omega \in S_*(A^{**})$) covered with p_G, i.e. with the states

$$\omega \in S_\mathscr{g} := [\varphi \in S(A): \varphi(p_G) = 1]\,. \tag{3.13}$$

To any $\omega \in S_\mathscr{g}$ corresponds the probabilistic measure $\mu_\mathscr{g}^\omega$ on the classical (generalized) phase space $\mathscr{g}^*$ determined by

$$\mu_\mathscr{g}^\omega(B) := \omega(E_\mathscr{g}(B)) \tag{3.14}$$

for Borel subsets B of $\mathscr{g}^*$. The states $\omega \in S_\mathscr{g}$ are called classical, cf. [3], and if $\mu_\mathscr{g}^\omega = \delta_F$ for some $F \in \mathscr{g}^*$ (δ_F is concentrated on the one-point set containing F) the state ω is called pure classical.

Let supp $E_\mathscr{g}$ be the minimal closed subset B of $\mathscr{g}^*$ such that $E_\mathscr{g}(B) = p_G$. Then $C_b(\text{supp}\, E_\mathscr{g} C)$ is the C^*-algebra of bounded continuous complex valued classical (intensive, macroscopic) observables of the large quantal system described by A and $\sigma(G)$. Let

$$C_{bs} := C_{bs}(\text{supp}\, E_\mathscr{g}, A) \tag{3.15}$$

be the C^*-algebra of A-valued norm-bounded s^*-continuous functions on supp $E_\mathscr{g}$ generated by functions $F \mapsto \sigma(h(F))\,(x)\ \ f(F)$, $f \in C_b$, $h \in C(\text{supp}\, E_\mathscr{g}, G)$, $x \in A$, where the s^*-topology is given by the seminorms on A:

$$x(\in A) \mapsto \sqrt{(\omega)}\,(x^*x)\,, \quad x \mapsto \sqrt{(\omega)}\,(xx^*) \quad \text{with} \quad \omega \in S_\mathscr{g}\,. \tag{3.16}$$

The 'classical' algebra C_b can be naturally identified with the subalgebra $C_{bs}(\text{supp}\, E_\mathscr{g}$, $\mathbb{C}\, \mathrm{id}_A)$ of C_{bs} and the algebra A is isomorphic to the algebra of constant functions in C_{bs}. Moreover, the algebra C_{bs} can be naturally embedded into $p_G A^{**}$ by the C^*-

isomorphism

$$\hat{f}(\in C_{bs}) \mapsto E_{\mathscr{g}}(\hat{f}) := \int \hat{f}(F)\, E_{\mathscr{g}}(\mathrm{d}F) \in p_G A^{**}\,, \tag{3.17}$$

where the integral is defined as a limit of integrals of step function approximations to the A-valued function $\hat{f}$. Denote

$$C := E_{\mathscr{g}}(C_{bs})\,, \quad \text{and} \quad C^J := E_{\mathscr{g}}(C^J_{bs}) \subset C\,, \tag{3.18}$$

where the images of $C^J_{bs} := C_{bs}(\operatorname{supp} E_{\mathscr{g}} A^J)$ (for finite $J \subset Z_+$) form a net of C^*-subalgebras of C endowing this 'algebra of observables of mean-field theories' with a quasilocal structure, cf. [12].

4. GENERALIZED MEAN-FIELD DYNAMICS

We shall introduce here the one parameter group τ^Q of time translations $\tau^Q_t \in$ *-aut $C(t \in \mathbb{R})$ corresponding to any differentiable function Q on $\mathscr{g}^*$. With the notation of the preceding sections, let

$$\hat{f}_t(F) := \sigma(g_Q^{-1}(t, F))\,(\hat{f}(\varphi^Q_t(F)))\,, \quad \hat{f} \in C_{bs}\,, \quad F \in \mathscr{g}^*\,, \quad t \in \mathbb{R}\,. \tag{4.1}$$

It can be shown that $\hat{f}_t \in C_{bs}$ and the mappings $\hat{f} \mapsto \hat{f}_t$ $(t \in \mathbb{R})$ form a one-parameter group of *-automorphisms of C_{bs}; the group property $\hat{f}_{t+s} = (\hat{f}_t)_s$ is a consequence of the cocycle identity (2.9) as well as of the group property of the classical flow φ^Q. Now we set:

$$\tau^Q_t(E_{\mathscr{g}}(\hat{f})) := E_{\mathscr{g}}(\hat{f}_t) \tag{4.2}$$

for all $\hat{f} \in C_{bs}$, $t \in \mathbb{R}$. The automorphism group $\sigma(G)$ leaves A^J invariant, cf. (3.4), hence τ^Q leaves invariant C^J $(J \subset Z_+)$. Let

$$N := E_{\mathscr{g}}(C_b) \quad \text{with} \quad C_b := C_{bs}(\operatorname{supp} E_{\mathscr{g}}, \mathbb{C}\, \mathrm{id}_A) \tag{4.3}$$

be the subalgebra of (bounded, continuous) classical observables, $N \subset Z p_G \subset Z$. For $\hat{f} \in C_b$, we have $\hat{f}(F) = \mathrm{id}_A f(F)$, where f is a $\mathbb{C}$-valued bounded continuous function on the (generalized) classical space, and $\hat{f}_t(F) = \mathrm{id}_A f(\varphi^Q_t(F))$, since the identity id_A of A is $\sigma(G)$-invariant. Hence

$$\tau^Q_t N = N\,, \quad \text{with} \quad \tau^Q_t(E_{\mathscr{g}}(f)) = E_{\mathscr{g}}(\varphi^{Q*}_t f)\,, \quad f \in C_b\,, \tag{4.4}$$

and we see that the 'mean-fields' time evolution restricted to the classical quantities reproduces the given classical flow φ^Q.

Let $x \in A$. Then

$$\tau^Q_t(x) = \int \sigma(g_Q^{-1}(t, F))\,(x)\, E_{\mathscr{g}}(\mathrm{d}F) \in C\,. \tag{4.5}$$

If the function $F \mapsto \sigma(g_Q^{-1}(t, F))\,(x)$ is not a constant on supp $E_{\mathscr{g}}$, the time-evolved element $\tau_t^Q(x)$ $(x \in A)$ does not belong to A: the algebra A of quasilocal observables is not, for a general 'mean-field evolution' τ^Q, time-invariant.

Let $\sigma(G) \subset$ *-aut A be a strongly continuous group, i.e. the functions $g \mapsto \sigma(g)(x)$ are norm-continuous for all $x \in A$. In this case, the subalgebra $C_b(\text{supp}\, E_g, A)$ of C_{bs} consisting of A-valued norm-bounded norm-continuous functions is invariant w.r.t. the mappings $\hat{f} \mapsto \hat{f}_t$ defined by (4.1), and this C^*-algebra is isomorphic to the (uniquely defined) C^*-tensor product $A \otimes N$. Hence, the 'algebra of observables of the mean-field theory' can be chosen in the form of the tensor product $A \otimes N$, if all the generators X_β $(\beta \in g)$ from (3.6) are bounded. This is the case of the spin systems, in which the considered type of mean-field interactions was traditionally used, cf. [3], [4]. The strong continuity of $\sigma(G)$ implies strong continuity of τ_R^Q. In the general case, however, τ^Q can be uniquely extended to a $\sigma(p_G A^{**}, p_G A^*)$-continuous one-parameter group of *-automorphisms of the von Neumann algebra $p_G A^{**}$ (containing the quasilocal algebra of the 'mean-field observables' C, cf. (3.17)).

We shall write an explicit expression for the derivation δ_Q — the generator of the group τ^Q. Let $X^J(\beta) := X^J_\beta := |J|\, X_{\beta J}$, cf. (3.7). Let $[\beta_j : j = 1, 2, \ldots, n]$ be an arbitrary basis of g, and $F_j := F(\beta_j)$ be coordinates of $F \in g^*$ in the dual basis. Let f be a function on g^* and $\delta_j f(F)$ is the value of the (possibility weak) partial derivative of f according to F_j in the point F. Then

$$X^J(\mathrm{d}_F Q) = \sum_{j=1}^{n} \delta_j\, Q(F)\, X^J(\beta_j)\,, \quad F \in g^*\,, \quad J \subset Z_+\,, \tag{4.6}$$

and the derivation can be written in the form:

$$\delta_Q(E_g(\hat{f})) = \int \Big(\mathrm{i}[X^J(\mathrm{d}_F Q), \hat{f}(F)] + \sum_{j=1}^{n} \delta_j \hat{f}(F)\, [Q, F_j]\,(F)\Big)\, E_g(\mathrm{d}F)\,, \tag{4.7}$$

where $\hat{f} \in C^J_{bs}$, cf. (3.18), is a 'sufficiently smooth' A^J-valued function on g^*, the first square bracket in (4.7) is the commutator of operators in the Hilbert space $p_G H_u$, cf. (3.5), and the second bracket $[Q, F_j]\,(F)$ denotes the classical Poisson bracket (2.2) of the functions Q and F_j: $F \mapsto F(\beta_j)$ on g^* in the point $F \in g^*$. We shall not investigate here the domain $D(\delta_Q)$ nor the exact meaning of the integral in (4.7). These questions are easily analysable in the case of bounded generators X_β, in which case $X^J(\mathrm{d}_F Q)$ are bounded. The operator-valued function

$$X_Q\colon (J; F) \mapsto X_Q(J, F) := X^J(\mathrm{d}_F Q) \tag{4.8}$$

can be called 'the Bogoliubov-Haag Hamiltonian', in accordance with the usual terminology, see e.g. [4]–[7]. The operator X_Q is useful in an analysis of thermodynamic properties of the mean-field models (these questions will be dealt with elsewhere) as well as in the calculation of the time-evolution: The eq. (2.9) for the function g_Q can be written in the form of differential (time-dependent Schrödinger) equation for the unitary family $U(g_Q(t, F))$:

$$\text{(4.9)} \qquad \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U(g_Q(t, F)) = X(\mathrm{d}_{F_t}Q)\, U(g_Q(t, F)), \quad F \in \mathscr{g}^*, \quad t \in \mathbb{R},$$

where $F_t := \varphi_t^Q(F)$, and all the operators are acting in the representation Hilbert space H of $U(G)$, as it was introduced at the beginning of sec. 3. The intiuitive picture of the dynamics of the considered models described in sec. 1 follows now easily from the aforesaid properties and connexions.

5. RELATION TO LOCAL HAMILTONIANS

We shall show here that the theory described in the preceding section can be obtained as the thermodynamic limit of local hamiltonian evolutions, at least in the case of the finite dimensional Hilbert spaces H and H_p ($p \in Z_+$) and of the polynomial Q (in a fixed coordinate basis in $\mathscr{g}^*$ dual to a given basis $[\beta_j: j = 1, 2, \ldots, n]$ in $\mathscr{g}$). In this case the generators $X(\beta) := X_\beta$ ($\beta \in \mathscr{g}$) are bounded. Let $X_j := X(\beta_j)$, $X_{jJ} := X_J(\beta_j)$, $X_j^J := X^J(\beta_j)$ and $X_{j\Pi} := X_{\beta_j \Pi} =: X_\Pi(\beta_j)$, cf. (3.8ii), for $j = 1, 2, \ldots$ $\ldots, n$, be the bounded selfadjoint operators defined in Sec. 3. Let

$$\text{(5.1)} \qquad Q^J := |J|\, Q(X_{1J}, X_{2J}, \ldots, X_{nJ})$$

be the selfadjoint operators (the 'local Hamiltonians') acting on the (finite dimensional) tensor product space

$$\text{(5.2)} \qquad H_J := \bigotimes_{p \in J} H_p, \quad \text{finite} \quad J \subset Z_+,$$

and obtained by substitution of X_{jJ} into the place of $F_j := F(\beta_j)$ in the polynomial $Q(F) = Q(F_1, F_2, \ldots, F_n)$ in which the order of the multiplication of variables is properly fixed (such that Q^J are symmetric for all J). We can consider $Q^J \in A^J$, since $X_{jJ} \in A^J$. The local time evolutions τ^J in the local algebras A^J are given by

$$\text{(5.3)} \qquad \tau_t^J(x) := \exp(\mathrm{i}tQ^J)\, x \exp(-\mathrm{i}tQ^J), \quad x \in A^J, \quad t \in \mathbb{R}.$$

We shall consider $A^J \subset A \subset A^{**}$ in the canonical way, cf. sec. 3 above the formula (3.6), and then we shall consider also $\tau_t^J \in$ *-aut A, or even (by keeping the same notation for the extensions of mappings) $\tau_t^J \in$ *-aut A^{**}. Since the center $\boldsymbol{Z}$ of A^{**} is elementwise τ^J-invariant, we can consider also $\tau_t^J \in$ *-aut $A^{**}p_G$. Let us denote

$$\text{(5.4)} \qquad B^J := A^J \cup [X_K(\beta): \beta \in \mathscr{g}, K \subset Z_+, K \text{ finite}].$$

Then there exists a positive number $r_J > 0$ such that the limits

$$\text{(5.5)} \qquad \tau_t^Q(x) := s^*\text{-}\lim_K \tau_t^K(x), \quad |t| < r_J, \quad x \in B^J,$$

exist in the $s^*(A^{**})p_G, S_{\mathscr{g}})$-topology given by the seminorms (3.16); this can be proved by estimates in the power expansions of $\tau_t^K(x)$. One can prove similarly the existence of

$$\tau_t^Q(E_\wp(f_\beta)) := s^* - \lim_K \tau_t^Q(X_{\beta K}) = E_\wp(\varphi_t^{Q*} f_\beta), \quad |t| < r_0, \tag{5.6}$$

where $r_0 := r_J$ for a one-point set J. Since the supp $E_\wp$ is compact, the elements $E_\wp(f_\beta)$ $(\beta \in \wp)$ generate the algebra N from (4.3). The mappings τ_t^Q from (5.5) and (5.6) leave the algebra $C^J \subset A^{**}p_G$ generated by $A^J p_G$ and by $N \subset Zp_G$ invariant and have a unique extension to an automorphism group of C^J. It can be shown that the C^*-algebra C^J is isomorphic to the tensor product $A^J \otimes N$. The mappings $\tau_t \in$ *-aut C^J $(t \in \mathbb{R}$, finite $J \subset Z_+)$ can be extended uniquely to the group $\tau^Q \subset$ $\subset$ *-aut C, where $C := A \otimes N$ is the quasilocal algebra generated by C^J $(J \subset Z_+)$.

The derivation of the group τ^Q can be written in the form

$$\delta_Q(x) = \mathrm{i} \sum_{j=1}^{n} E_\wp(\delta_j Q) \, [X_j^J, c] \quad \text{for} \quad x \in A^J, \tag{5.7}$$

$$\delta_Q(E_\wp(f)) = E_\wp([Q, f]) \quad \text{for} \quad f \in C^1(\wp^*). \tag{5.8}$$

These expressions coincide with (4.7) and determine the group τ^Q, as follows from the general theory of automorphism groups, cf. [12]. This proves that the conventional notion of 'mean-field theories' determined by the net of local Hamiltonians Q^J coincides with the definition of the dynamics given in our sec. 4. Detailed proofs of all the assertions of this paper will be published elsewhere, cf., e.g., [13].

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